On Quantum Ambiguity and Potential Exponential Computational Speed-Ups to Solving Dynamic Asset Pricing Models*

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Abstract

We formulate quantum computing solutions to a large class of dynamic nonlinear asset pricing models using algorithms, in theory exponentially more efficient than classical ones, which leverage the quantum properties of superposition and entanglement. The equilibrium asset pricing solution is a quantum state. We introduce quantum decision-theoretic foundations of ambiguity and model/parameter uncertainty to deal with model selection.

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1 Introduction

Quantum computing (QC) is a paradigm shift in computer science that aims to use the principles of quantum mechanics to solve computational problems that are beyond the capabilities of classical computers. The current state of quantum computing is promising but still in a nascent stage. In the foreseeable future, quantum hardware will become mainstream, in particular in a hybrid combination with classical computing. In anticipation of this happening, this paper makes two types of contributions. First, we propose a new class of numerical solutions to dynamic asset pricing models. The equilibrium solution comes in the form of a quantum state, and therefore not the usual output one gets from current solution methods. The second contribution of the paper is to take advantage of the quantum state solution and introduce both a theoretical and practical framework for econometricians who cope with model selection in the face of misspecification and ambiguity, assembling elements from (a) quantum computing and measurement, (b) statistical decision theory and (c) quantum models of decision-making. In particular, we obtain a decomposition yielding new insights into model selection, separating classical and quantum ambiguity, with the latter harnessing new capabilities of quantum hardware. The notions of quantum ambiguity, originated in the decision theory literature to solve the Ellsberg and other behavioral paradoxes, are novel to the econometrics literature.

First, we discuss the numerical solution part of the paper. We are interested in the potential opportunities quantum computing can bring to advance fundamental research in asset pricing. More specifically, we are interested in solving an equation of interest in dynamic asset pricing models:

$$\nu(x_t) = \int \psi(x_{t+1}, x_t)\nu(x_{t+1})f(x_{t+1}|x_t)dx_{t+1} + g(x_t). \tag{1.1}$$

The equation, also known as Fredholm equations of the second type, involves functions $\psi(\cdot,\cdot)$ and $g(\cdot)$ which depend on economic determinants, including the stochastic discount factor (SDF). We are interested in characterizing $\nu(\cdot)$, the solution to a dynamic asset pricing problem, such as the equilibribum price-dividend ratio.[1]

We can rewrite equation (1.1) as follows $\nu = \mathcal{T}[\nu] + g$, where $\mathcal{T}[\nu]$ is the operator defined by

[1]We deliberately start from a discrete time setting while keeping in mind that continuous time asset pricing models either have discrete time analogues to equation (1.1) or some discretization provides is a workable approximation.

the integral term, which under suitable regularity conditions is invertible, i.e. the operator $[I - \mathcal{T}]^{-1}$ exists, where I denotes the identity operator.[2] This yields the solution $\nu = [I - \mathcal{T}]^{-1} g$. The quadrature approach of Tauchen and Hussey (1991) amounts to proposing an approximate solution $\nu_N = [I - \mathcal{T}_N]^{-1} b_N$ using $\mathcal{T}_N$ instead of $\mathcal{T}$ with the former a quadrature-based discrete version close to the latter. The approximation error diminishes as $N \to \infty$.

Practical implementations often require inverting a high-dimensional matrix. In this paper we propose to use QC algorithms to handle these high-dimensional computations. With classical computers the time complexity grows at best linearly in N. More precisely, a classical computer using the conjugate gradient method requires $O(Ns\kappa \log(1/\varepsilon))$ running time, where s is a measure of sparsity, κ a conditioning number and ε the accuracy of the approximation.[3] The QC algorithm has a running time complexity of $O(\log_2(N)s^2\kappa^2/\varepsilon)$, an exponential speed-up in the size of the system (i.e. QC algorithms have a worse runtime than conjugate gradient descent in terms of the error and condition number, but are exponentially better in terms of the dimension of the system). One crucial caveat to keep in mind is that QC hardware is not as reliable as classical computing, but the fidelity of QC is steadily improving as innovations in hardware are implemented.

The potential of exponential speed-ups is one thing. Arguably of more importance is the fact that quantum computers allow us to entertain and implement notions of ambiguity beyond those primarily used in the econometrics literature. Solutions to asset pricing models, using the methods proposed in this paper, are represented by quantum states. This comes with its own challenges but also offers new exciting opportunities. In particular, we need to draw attention to the fact that measuring output generated by QC algorithms is not as straightforward as it is with classical computers. When a measurement is performed on a quantum system, it "collapses" to a specific state with certain probabilities. In physics, measurement refers to physical instruments used in lab experiments to measure quantum states. Mathematically, the process of measurement is modeled via unitary operators defined on tensor product of Hilbert spaces representing quantum states. The innovation in this paper is that

[2] We defer to Section 2 the formal discussion regarding the conditions for a proper integral and associated operator $\nu = \mathcal{T}[\nu] + g$ in equation (1.1).

[3] The condition number of the matrix is defined as the ratio of the smallest and the largest eigenvalue and sparsity s measures the max non-zero entries per row or column. Time complexity $O(\cdot)$ denotes an upper bound on the worst case of a problem. Inverting an N-dimensional square symmetric matrix via elementary methods (such as Gaussian elimination) theoretically takes $O(N^3)$ amount of time, i.e. has a polynomial rate of the third order. More sophisticated algorithms, using the conjugate gradient method achieve linear rates.

we take a decision-theoretic approach to quantum state measurement amenable to econometric model specification.

Axiomatic Foundations of Decision Making	$\Rightarrow$	**Decision-theoretic Foundations of Statistical Inference**
Savage (1954), Anscombe and Aumann (1963), Gilboa and Schmeidler (1989), among others		Cerreia-Vioglio et al. (2013), Denti and Pomatto (2022), Hansen and Sargent (2023), among others
$\Downarrow$		$\Downarrow$
Quantum Behavioral Models of Decision Making	$\Rightarrow$	**Quantum Statistical Inference**
Aerts (2009), Yukalov and Sornette (2010), among others		This paper

Table 1.1: Decision-theoretic Foundations of Quantum Statistical Inference

In Table 1.1 we provide some key references to three different areas of research which feed into our QC approach to solving asset pricing models. At the outset it should be noted that our suggested approach can be applied to a broader set of research questions across different fields beyond the specific application to asset pricing. Our approach is much inspired by the recent work by Hansen and Sargent (2023) who explore linkages between model specification, risk/uncertainty and ambiguity, and connecting decision-theoretic foundations of statistical decision theory with econometric model specification. The upper left and right entries in Table 1.1 refer to the recent work by Hansen and Sargent (2023) and its decision-theoretic antecedents by Savage (1954), Anscombe and Aumann (1963), Gilboa and Schmeidler (1989), among others. Because the equilibrium asset pricing solution is a quantum state, this means that we look at states and statistical decisions that are more complex with von Neumann's (1932) quantum probability foundations instead of Kolmogorov's (1933) axiomatic foundations of probability (sometimes called classical) on which the decision theory - used by Hansen and Sargent (2023) and the many papers cited by them - relies on. In particular, our approach is inspired by quantum behavioral models of decision-making appearing in the lower left corner of Table 1.1. We relate to a literature which proposed von-Neuman Morgenstern expected utility solutions to the Ellsberg (1961) paradox and other behavioral anomalies, and thus providing rational decision making solutions in ambiguous settings.

This paper is part of a small but growing literature on quantum computing applications in eco-

nomics and finance. Orus et al. (2019) and Herman et al. (2022) survey the burgeoning literature and Hull et al. (2024) offers an introduction to quantum technology that is specifically tailored to economists. Regarding financial applications, the focus has been primarily on financial derivatives and tail risk because QC algorithms can deliver quadratic speed-ups compared to Monte Carlo simulation methods.[4] Fernández-Villaverde and Hull (2022) propose a quantum computing approach to solving dynamic programming problems and show that current vintage hardware is capable of achieving an order-of-magnitude speed-up in solving the real business cycle model over benchmarks in the literature. It is important to emphasize that implementing the methods proposed in our paper are challenging with the currently available QC hardware. Hence, our paper is not (yet) about showing QC speed-ups currently achievable. Instead, we expect the steady developments in hardware to make the research methods we propose feasible in the near future. We therefore use classical methods to mimic QC hardware as a proof of concept instead.

The paper is structured as follows. In Section 2 we start with describing the class of asset pricing models and summarize the quadrature approach of Tauchen and Hussey (1991) to solving such models. In Section 3 we present the QC algorithms. Section 4 covers the decision-theoretic foundations of QC measurement followed by Section 5 discussing quantum versus classical ambiguity. Section 6 reports on an empirical asset pricing application. Conclusions appear in Section 7.

Notation: The Hadamard product between two same-sized matrices $A = [a_{ij}]_{i,j} \in \mathbb{R}^{N_1 \times N_2}$ and $B\,[b_{ij}]_{i,j} \in \mathbb{R}^{N_1 \times N_2}$ is denoted by $A \circ B \in \mathbb{R}^{N_1 \times N_2}$ corresponds to the element-wise matrix product, i.e. $A \circ B := [a_{ij} b_{ij}]_{i,j}$. The computational part of the paper adopts the Dirac (1939) bra-ket notation used in quantum mechanics, as well as the quantum computation and information literature (see Nielsen and Chuang (2010) for a standard textbook reference or the Appendix to Morrell et al. (2021) which provides a quick summary focused on the concepts used in our paper).[5] The basic building block is a Hilbert space $\mathbb{H}$, which is a complete normed vector space over $\mathbb{C}$ with inner product denoted $\langle \cdot | \cdot \rangle$:

[4]There are many applications, including Orus et al. (2019), Woerner and Egger (2019), Chakrabarti et al. (2020), Stamatopoulos et al. (2020) for Black-Scholes type models with exotic option pricing. See Kaneko et al. (2022), Vazquez and Woerner (2021), Ghysels et al. (2023), for option pricing models with stochastic volatility. QC also produces potential quadratic speed-ups in computing quantiles and are therefore applied to credit risk models, see Egger et al. (2020) and Ghysels et al. (2023).

[5]The Dirac bra-ket notation uses $|u_i\rangle$ (called a ket) for the complex column vector u_i and $\langle u_i|$ (a bra) is written for its adjoint, the row vector containing the complex conjugates of its elements, with the complex conjugate written as $\overline{u}_i$. The quadratic form $\overline{u}_i A u_i$ is then written as $\langle u_i | A | u_i \rangle$. The notation allows one to distinguish numbers from matrices, as in $\langle u_i | u_i \rangle$ versus $|u_i\rangle \langle u_i|$, and to specify vectors through labels or descriptions as in $|\text{Model 1}\rangle$.

$\mathbb{H} \times \mathbb{H} \to \mathbb{C}$. Of particular interest to us is the tensor product space $\bigotimes_{i=1}^{n} \mathbb{H}$ and orthonormal basis which will be denoted $|u_i\rangle$ for $i = 0, \ldots, n-1$.

2 Dynamic Asset Pricing Models

We start in subsection 2.1 with a review of the approach advanced by Tauchen and Hussey (1991) to solving equation (1.1) using numerical quadrature methods. In the next subsection we explore issues regarding model specification and ambiguity.

2.1 A Review of Quadrature Methods

We start with a probability space, namely, a triple $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is a collection of discrete time ($t \in \mathbb{N}$) infinite sequences in $\mathbb{R}^{n_y}$, $\mathcal{F}$ is the smallest sigma algebra of events in Ω and $\mathcal{P}$ assigns probabilities to events. The state of the economy is described by the n_y-dimensional stationary stochastic process $\{y_t \in \mathbb{R}^{n_y} : t = 1, \ldots, \infty\}$. Also of interest is the sigma filtration $\mathcal{F}_t$ and associated $\mathcal{P}_t$ pertaining to events up to time t. We focus on a single asset with ex-dividend price p_t at time time t and future dividend stream $\{d_{t+k}, k = 1, \ldots, \infty\}$. Denote the price-dividend ratio by $\nu_t = p_t/d_t$. Given a representative agent's time t stochastic discount factor (SDF) $m(y_{t+1}, y_t)$, we have:

$$\nu_t = \mathbb{E}_t \left[(1 + \nu_{t+1}) h_{t+1} m(y_{t+1}, y_t) \right] \tag{2.1}$$

where $h_{t+1} = d_{t+1}/d_t = h(y_{t+1})$ is the dividend growth and $\mathbb{E}_t[\cdot]$ the conditional expectation given the $\mathcal{F}_t$ filtration. We restrict our attention to stationary Markov processes with conditional distribution of y_t given its past $x_{t-1} = \{y_{t-1}\}$ given by $f(y_t|x_{t-1})$. We can write equation (2.1) in integral form as follows:

$$\begin{aligned}
\nu(x) &= \int [1 + \nu(y)] \psi(y, x) f(y|x) dy \\
&= \int [1 + \nu(y)] \psi(y, x) \frac{f(y|x)}{\omega(y)} \omega(y) dy \qquad \text{for some } \omega(y) > 0 \\
\underbrace{\nu(x)}_{\nu} &= \underbrace{\int \nu(y) \psi(y, x) \frac{f(y|x)}{\omega(y)} \omega(y) dy}_{\mathcal{T}[\nu]} + \underbrace{\int \psi(y, x) \frac{f(y|x)}{\omega(y)} \omega(y) dy}_{g}
\end{aligned} \tag{2.2}$$

where $\nu(x) : \mathbb{R}^{n_y} \to \mathbb{R}$ is the price-dividend ratio as a function of the current state, while $\psi(y, x) = h(y) \times m(y, x)$ with $h(y)$ the dividend growth as function of the future state and $m(y, x)$ the SDF.

REMARK 2.1. *Estimation and inference of the asset pricing model appearing in equation (2.2) require a parametric specification for the functions $\lambda(y, x)$, $f(y|x)$, and $\omega(y)$.*[6] *We postpone the characterization of parameter spaces until Section 2.2.*

From the last expression in equation (2.2) we define the integral operator $\mathcal{T}$ as follows:

$$\nu = \mathcal{T}[\nu] + g. \tag{2.3}$$

REMARK 2.2. *Equations (2.1) through (2.3) involve integration of functions $\lambda(y, x) : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \to \mathbb{R}$, $f(y|x) : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \to \mathbb{R}$, and $\omega(y) : \mathbb{R}^{n_y} \to \mathbb{R}$. We will refrain here from stating formal assumptions to guarantee the integrals are well defined and the implied integral operator $\mathcal{T}$ is bounded. We refer to Tauchen and Hussey (1991) for a formal discussion of regularity conditions.*

Next, we impose a straightforward (for our particular context) technical assumption regarding $h(y)$, namely since $h_{t+1} = d_{t+1}/d_t = h(y_{t+1})$, we exclude negative dividends with the following assumption:

ASSUMPTION 2.1. *The function $h : \mathbb{R}^{n_y} \to \mathbb{R}$, is non-negative.*

And because $\psi(y, x) = h(y) \times m(y, x)$ with the SDF being positive by no-arbitrage conditions, we also have by implication of the above assumption that $\psi(y, x) \geq 0 \; \forall \; x$ and y.

The integral (operator) equation (2.3) will be approximated by the N-point quadrature rule $\nu_N = \mathcal{T}_N \nu_N + b_N$. To that end, let $\bar{y}_k$ and w_k for $k = 1, \ldots, N$, be the abscissa and weights for an N-point quadrature rule for the density $\omega(y)$. A quadrature rule can be viewed as a discrete probability model that approximates the density ω, and in the case of Gauss quadrature rules these approximations are close to minimum norm rules, yielding for $\psi_{jk} = \psi(\bar{y}_k, \bar{y}_j)$:

$$\begin{aligned} \bar{\nu}_{Nj} &= \sum_{k=1}^{N} [1 + \bar{\nu}_{Nk}] \, \psi_{jk} \pi_{jk}^{N} \\ &= \sum_{k=1}^{N} [\bar{\nu}_{Nk}] \, \psi_{jk} \pi_{jk}^{N} + \sum_{k=1}^{N} [1] \, \psi_{jk} \pi_{jk}^{N} \quad j = 1, \ldots, N \end{aligned} \tag{2.4}$$

[6] In principle we can also think of non-parametric approaches, but these are beyond the scope of the current paper.

where $\pi^N_{jk} = \pi^N_k(\bar{y}_j)$ for $\pi^N_k(x) = [f(\bar{y}_k|x)/(s(x)\omega(\bar{y}_k))]w_k$ and $s(x) = \sum_{j=1}^N [f(\bar{y}_j|x)/\omega(\bar{y}_j)]w_i$. Therefore:

$$\underbrace{\begin{bmatrix} \bar{\nu}_{N1} \\ \vdots \\ \bar{\nu}_{Nj} \\ \vdots \\ \bar{\nu}_{NN} \end{bmatrix}}_{\bar{\nu}_N} = \underbrace{\begin{bmatrix} \psi_{11}\pi^N_{11} & \dots & \psi_{1N}\pi^N_{1N} \\ & & \\ \vdots & \ddots & \vdots \\ & & \\ \psi_{N1}\pi^N_{N1} & \dots & \psi_{NN}\pi^N_{NN} \end{bmatrix} \begin{bmatrix} \bar{\nu}_{N1} \\ \vdots \\ \bar{\nu}_{Nj} \\ \vdots \\ \bar{\nu}_{NN} \end{bmatrix}}_{\mathcal{T}_N \bar{\nu}_N} + \underbrace{\begin{bmatrix} \sum_{k=1}^N \psi_{1k}\pi^N_{1k} \\ \vdots \\ \sum_{k=1}^N \psi_{jk}\pi^N_{jk} \\ \vdots \\ \sum_{k=1}^N \psi_{Nk}\pi^N_{Nk} \end{bmatrix}}_{b_N}. \tag{2.5}$$

REMARK 2.3. *Under suitable regularity conditions, see e.g. Davis and Rabinowitz (2014) and Tauchen and Hussey (1991), the approximate solution $\nu_N = [I - \mathcal{T}_N]^{-1} b_N$ convergences to the function $\nu(x) \in \mathbb{R}$ solving the integral equation (2.2). We refrain here from stating formal assumptions as the main focus of the paper is discrete state approximations. For our analysis we keep N finite for a number of technical reasons. Dealing with infinite dimensional settings is left for future research, as explained in the Conclusion Section 7.*

To proceed, recall that $\psi(y, x) = h(y) \times m(y, x)$, which motivates defining the following matrices $\Psi_N := [\psi_{ij}]_{i,j=1,\dots,N}$, $\mathcal{M}_N := [m_{ij}]_{i,j=1,\dots,N}$, $\Pi_N := [\pi_{ij}]_{i,j=1,\dots,N}$, $\mathcal{H}_N = [[h_i]_{i=1,\dots,N} \times \mathbf{1}'_N]$ where $\mathbf{1}_N$ is a $N \times 1$ vector of ones, and finally $\Psi_N = \mathcal{H}_N \circ \mathcal{M}_N$.

ASSUMPTION 2.2. *The $N \times N$ matrix $[I_N - \Psi_N \circ \Pi_N]$ is full rank and therefore invertible.*

Under Assumption 2.2 the following is an approximate solution to the fundamental pricing equation (1.1):

$$\bar{\nu}_N = [I_N - \Psi_N \circ \Pi_N]^{-1} b_N \quad \text{or equivalently} \quad \bar{\nu}_N = [I_N - \mathcal{H}_N \circ \mathcal{M}_N \circ \Pi_N]^{-1} b_N \tag{2.6}$$

which can be viewed as the solution to the asset pricing equations where the law of motion of the state vector is a discrete Markov chain with the N quadrature abscissa $\bar{y}_j$ as states and transition probabilities $\pi^N_{jk} = \Pr(y_t = \bar{y}_k \mid y_{t-1} = \bar{y}_j)$. Put differently, we can think of the discrete Markov chain as a proxy for

$f(y|x)$ in equation (2.2) and the Nystrom extension of the solution to the entire domain of x is

$$\tilde{\nu}_N(x) = \sum_{k=1}^{N} [1 + \bar{\nu}_{Nk}] \, \psi(\bar{y}_k, x) \pi_k^N(x) \qquad x \in \mathbb{R}^M. \tag{2.7}$$

Inspecting equation (2.5) we can rewrite it as follows: $\bar{\nu}_N = [\Psi_N \circ \Pi_N]\bar{\nu}_N + b_N$, or equivalently $\bar{\nu}_N = [\mathcal{H}_N \circ \mathcal{M}_N \circ \Pi_N]\bar{\nu}_N + b_N$. Looking at $\mathcal{H}_N$, $\mathcal{M}_N$ and Π_N separately has the advantage that we can distinguish the role of dividend growth from that of the stochastic discount factors and/or the transition probabilities. By the same token we can simply use Ψ_N as the discounted dividend growth process as a joint component.

REMARK 2.4. *In equation (2.2) we presented a quadrature rule where the weights w_k only depend on an unconditional density ω, as suggested by Tauchen and Hussey (1991). Farmer and Toda (2017) provide a method for accurately discretizing general Markov processes by matching low order moments of the conditional distributions using maximum entropy. We use their approach.*

The next assumption is needed to rely on the Markov chain convergence theorem, see e.g. Meyn and Tweedie (1993).

ASSUMPTION 2.3. *The discrete state Markov chain characterized by transition density Π_N is aperiodic and Harris recurrent.*

Under Assumption 2.3 we can characterize the ergodic distribution denoted as π_N^e as the eigenvector corrresponding to the eigenvalue 1 of the matrix Π_N.

REMARK 2.5. *So far we focused on dealing with solving equation (1.1) and as we explained in the Introduction, the QC solution methods involve finding eigenvalues and eigenvectors of $[I_N - \Psi_N \circ \Pi_N]$. It should be noted that much of our analysis also applies to an arguably simpler problem:*

$$\rho\phi(x_t) = \int m(x_{t+1}, x_t)\phi(x_{t+1}) f(x_{t+1}|x_t) dx_{t+1}, \tag{2.8}$$

which pertains to Hansen and Scheinkman (2009) who link the permanent-transitory decomposition of stochastic discount factors (SDF) $m(x_{t+1}, x_t)$ in Markovian settings, with a M-dimensional driving process x_t and the conditional density $f(x_{t+1}|x_t)$, to a Perron-Frobenius eigenfunction problem. The

eigenvalue ρ determines the average yield on long-horizon payoffs and the eigenfunction $\phi(\cdot)$ is positive and characterizes dependence of the price of long-horizon payoffs on the Markov state. Using a N-point discretization we can use standard Perron-Frobenius theory to solve the eigenvector $\rho_N\,\phi_N = [\mathcal{M}_N \circ \Pi_N]\,\phi_N$. More specifically, for a discrete time Markov chain, see Example 6.1 of Hansen and Scheinkman (2009), we can write:

$$\rho_N \begin{bmatrix} \phi_{N1} \\ \vdots \\ \phi_{Nj} \\ \vdots \\ \phi_{NN} \end{bmatrix} = \begin{bmatrix} m_{11}\pi_{11}^N & \dots & m_{1N}\pi_{1N}^N \\ & & \\ \vdots & \ddots & \vdots \\ & & \\ m_{N1}\pi_{N1}^N & \dots & m_{NN}\pi_{NN}^N \end{bmatrix} \begin{bmatrix} \phi_{N1} \\ \vdots \\ \phi_{Nj} \\ \vdots \\ \phi_{NN} \end{bmatrix}, \tag{2.9}$$

where the principal eigenvector characterizes the long-run risk associated with the SDF. No-arbitrage implies $\mathcal{M}_N > 0$ and for Π_N we can make the following observation. As noted by Hansen and Scheinkman (2009) in their Example 6.1, a principal eigenvector is found by finding an eigenvector of $[\mathcal{M}_N \circ \Pi_N]$ with strictly positive entries. Standard Perron-Frobenius theory implies that if the chain is irreducible (cfr. Assumption 2.3), since the multiplicative functional is strictly positive, there is such an eigenvector which is unique up to scale.[7]

Finally, we introduce some generic notation for equation (2.6) and a variation of it more suitable for quantum computing. Starting with the former, we will use the following generic notation:

$$\mathcal{A}_N\bar{\nu}_N = b_N \quad \text{with } \mathcal{A}_N = [I_N - \mathcal{T}_N] = [I_N - \mathcal{H}_N \circ \mathcal{M}_N \circ \Pi_N] \quad \text{and therefore } \bar{\nu}_N = \mathcal{A}_N^{-1} b_N. \tag{2.10}$$

Quantum mechanics involve unitary operator applications to unit-norm vectors in Hilbert spaces. What that in mind, we will rewrite b_N as: $b_N \equiv \text{diag}(\sqrt{N}\sum_{k=1}^N \psi_{1k}\pi_{1k}^N, \dots, \sqrt{N}\sum_{k=1}^N \psi_{Nk}\pi_{Nk}^N) \times \iota_N \equiv \mathcal{B}_N\iota_N$, with $\mathcal{B}_N$ = $\text{diag}(\sqrt{N}\sum_{k=1}^N \psi_{1k}\pi_{1k}^N, \dots, \sqrt{N}\sum_{k=1}^N \psi_{Nk}\pi_{Nk}^N)$ and ι_N is a unit-norm vector in a N-dimensional space. We can rewrite equation (2.10) as:

$$(\mathcal{B}_N^{-1}\mathcal{A}_N)\bar{\nu}_N = \iota_N \quad \text{and therefore } \bar{\nu}_N = \mathcal{C}_N^{-1}\iota_N \quad \text{with } \mathcal{C}_N = (\mathcal{B}_N^{-1}\mathcal{A}_N). \tag{2.11}$$

[7] The convergence of ρ_N and ϕ_N to respectively ρ and ϕ in equation (2.8) - assuming their existence and uniqueness - is further discussed in Hansen and Scheinkman (2009).

Note that $\mathcal{C}_N = \mathcal{B}_N^{-1}\mathcal{A}_N$ encodes all the information about the model and we can therefore focus on $\mathcal{C}_N$ when we study multiple models.

2.2 Ambiguity, Parameter Uncertainty and Risk

Econometricians worry about model specification errors. In a series of papers Lars Hansen and Thomas Sargent have developed a framework to formalize the links between risk, ambiguity (about Bayesian priors), and misspecification (of the assumed model) and their decision-theoretic, robust control, and statistical foundations (see Hansen and Sargent (2023) and references therein).

The goal of our paper is to cast the solution of dynamic asset pricing models in the context of QC. Given the potential speed-ups we will explore multiple model specifications either due to parameter uncertainty or ambiguity about the model or both. In the remainder of the paper we will use the terms ambiguity and douts interchangeably.

We rely on a framework suggested by Hansen and Sargent (2023) who align definitions of statistical models, uncertainty, and ambiguity with ideas from decision theories that build on representations of subjective and objective uncertainties articulated by Anscombe and Aumann (1963). They adopt a version of Anscombe and Aumann (1963) (henceforth AA) where model parameters are *states* with $(\Theta, \mathfrak{G})$ a measurable space of potential states. Let $\mathscr{P}$ be the set of probability measures over states and for each $p \in \mathscr{P}$, $(\Theta, \mathfrak{G}, p)$ is a probability space.[8]

For each parameter vector $\theta \in \Theta$, we define the vector $\bar{\nu}_N(\theta) \in \mathbb{R}^N$ of model-implied price-dividend ratios. Moreover, using the definitions in equation (2.10), we have $\mathcal{A}_N(\theta)\ \bar{\nu}_N(\theta) = b_N(\theta)$ with $\mathcal{A}_N(\theta) = [I_N - \mathcal{T}_N(\theta)] = [I_N - \mathcal{H}_N(\theta) \circ \mathcal{M}_N(\theta) \circ \Pi_N(\theta)]$ and therefore $\bar{\nu}_N(\theta) = \mathcal{A}_N^{-1}(\theta) b_N(\theta)$. An alternative characterization is: $\mathcal{C}_N(\theta) = \mathcal{B}_N^{-1}(\theta)\mathcal{A}_N(\theta)$ which also encodes all the information about the model for a given parameter vector.

REMARK 2.6. *We do not exactly follow the setup of Hansen and Sargent (2023), but instead fashion it to our specific model of interest, namely solving dynamic asset pricing models of the type appearing in equation (1.1). There are some non-trivial deviations, notably in the assumption below, which will be further discussed.*

[8] See also the summary in Table 4.1 appearing in the next section.

To avoid additional notation we assume that the price-dividends in $\bar{\nu}_N(\theta) \in \mathbb{R}^N$ are sorted from low to high such that we can associate the solution of a model, given θ, with a distribution function over possible *prizes* in the Anscombe and Aumann (1963) sense. Namely:

ASSUMPTION 2.4. *The equilibrium price-dividends in $\bar{\nu}_N(\theta) \in \mathbb{R}^N$ represent Anscombe and Aumann (1963) prizes. Moreover, for each $\theta \in \Theta$ there is a unique $\bar{\nu}_N(\theta)$ and for any two θ_1 and $\theta_2 \in \Theta$, $\bar{\nu}_N(\theta_1) \neq \bar{\nu}_N(\theta_2)$. Hence, the parameter vector relates bijectively to prizes.*

Assumption 2.4 identifies a unique N-point vector with each parameter θ. In the next section, we will further explore the decision-theoretic foundations, adopting a quantum computational and informational approach. This will highlight the unique features of QC, and also showcase opportunities that come with it.

We conclude with a discussion of multiple models as it will be helpful in the analysis of misspecification risk and ambiguity. We use the notation N_D for dimension of the discrete states and N_M as the number of models. Note that N_D is determined by a quadrature rule. For simplicity we assume that N_D is the same across the N_M different models. To address the issue of multiple models we start from a situation where m = 1, ..., N_M plausible models are put forward and each model m is associated with a measurable space of potential states $(\Theta_m, \mathfrak{G}_m)$. Let $\mathscr{P}_m$ be the set of probability measures over states and for each $p_m \in \mathscr{P}_m$, (Θ_m, $\mathfrak{G}_m$, p_m) is a probability space. This means we have a collection $\{(\mathcal{A}^m_{N_D}(\theta_m), b^m_{N_D}(\theta_m)) : m = 1, \ldots, N_M\}$ or $\{\mathcal{C}^m_{N_D}(\theta_m) : m = 1, \ldots, N_M\}$ (cfr. equation (2.11)) and associated solutions $\{\bar{\nu}^m_{N_D}(\theta_m) : m = 1, \ldots, N_M\}$.

The solution $\bar{\nu}^m_{N_D}(\theta_m)$ has several potential sources of specification error since $\mathcal{A}^m_{N_D}(\theta_m)$ =$[I_N - \mathcal{H}^m_{N_D}(\theta_m) \circ \mathcal{M}^m_{N_D}(\theta_m) \circ \Pi^m_{N_D}(\theta_m)]$. Namely, we may have concerns about $\Pi^m_{N_D}(\theta_m)$ in particular as the persistence of the state process might be in doubt. Similarly, the stochastic discount factor $\mathcal{M}^m_{N_D}(\theta_m)$ could be a source of model error. Lastly, and probably least likely, the payoff function embedded in $\mathcal{H}^m_{N_D}(\theta_m)$ might be prone to mistakes. Note that by implication $b^m_{N_D}(\theta_m)$ is affected by these uncertainties as well.

REMARK 2.7. *Anscombe and Aumann (1963) distinguish "lotteries" governed by known, i.e. objective, probabilities from "horse races" with unknown (subjective) probabilities. Hansen and Sargent (2023) adopt this framework in the context of statistical inference. We follow a similar approach,*

though again different and tailored to the specific application we have mind. To that end, we will sometimes split the parameter vector $\theta \in \Theta$, with $\theta \equiv (\theta_L, \theta_H) \in \Theta_L \times \Theta_H$ with the former governed by some objective probability distribution, whereas the latter by a subjective one. Case in point, specifically of interest in the context of our application is $\mathcal{A}_N(\theta) = [I_N - \mathcal{H}_N(\theta_L) \circ \mathcal{M}_N(\theta_H) \circ \Pi_N(\theta_L)]$, meaning the econometrician has some objective distribution to assess the model for state dynamics, but has potential doubts about the SDF. Of course, we can also consider the two polar cases $\theta \equiv \theta_L$ and $\theta \equiv \theta_H$.

3 Quantum Computational Solutions for Asset Pricing Models

Quantum algorithms for solving linear systems of equations are named HHL, after the three authors Aram Harrow, Avinatan Hassidim and Seth Lloyd, who introduced them (see Harrow et al. (2009)). The HHL algorithm leverages the quantum properties of superposition and entanglement to solve certain types of linear systems more efficiently than classical algorithms. The HHL algorithm focuses on solving systems of linear equations of the form $A\,|x\rangle = |b\rangle$, where A is an $n \times n$ matrix and using the Dirac notation for $|x\rangle$ and $|b\rangle$, which means that they will refer to quantum states, as further explained later. While it has theoretically been shown by Harrow et al. (2009) that their algorithm provides exponential speed-ups in solving linear systems, it has been deemed challenging in terms of implementation with current NISQ era hardware, where NISQ stands for Noisy Intermediate-Scale Quantum, which refers to a class of quantum computing devices that are characterized by intermediate-scale quantum processors and a high level of noise in their quantum operations. Implementing HHL on current hardware is challenging, which is why a number of modifications have been suggested to make it more amenable to NISQ hardware. These includes the Hybrid-HHL of Lee et al. (2019) and Yalovetzky et al. (2022) and the Enhanced Hybrid-HHL of Morgan et al. (2024). Even with these modifications, using HHL to invert a system of dimension 4 by 4 is challenging due to hardware noise. We therefore classically mimic a hypothetical HHL implementation without hardware noise nor algorithmic error.

First, the intuition regarding the exponential speed-up. Classical computers use bits to represent information, quantum computers use qubits (quantum bits), which can exist in multiple states at once

to perform calculations. Let's look at approximate solution $\nu_N = [I - \mathcal{T}_N]^{-1} b_N$. It takes n_b qubits to store the information in a vector such as b of size $N = 2^{n_b}$ (assuming n_b is integer). The vector b is translated to a quantum state and linear algebraic operations, such as applying $[I - \mathcal{T}_N]^{-1}$, amount to manipulation of the information in the n_b qubits using quantum mechanics principles which are formally described as unitary operators defined on n_b-dimensional tensor products of Hilbert spaces. Hence, we stay within the n_b-dimensional setup going through motions which amount to matrix inversion. These motions have a time complexity proportional to $\log_2 N$. While this is certainly an oversimplification, it is intuitively why there is a potential exponential speed-up.

Second we discuss why it matters. Suppose we have a four dimensional state variable y_t. One can think of monthly data, say aggregate consumption growth, and we take current month and three lags which yields the four dimensional state process, i.e. $n_y = 4$. In addition, suppose we take a five point quadrature rule. That means $N = 5^4 = 625$ whereas $\log_2 N < 10$. When we add another series with four lags and move to a nine point quadrature rule we have $N = 9^8 = 43,046,721$ whereas $\log_2 N < 26$. Even though this is only a small-scale example with two economic driving processes each having three lags, the numerical computations grow quickly beyond our reach or are too costly with classical computing.When we add model uncertainty, the above arguments are even more of critical importance. For example, Hansen (2007) considers four univariate AR model parameter configurations or submodels, where a submodel is a collection of states for which there is no chance of leaving that collection. Each of the four models have a 100-point discretitzed Markov chain, adding up to a 400 state Markov chain to approximate a model selection problem or estimation problem for investors. While this is a simple example, it clearly illustrates how robustness concerns amplify the computational burden, in the case of classical computers it grows linear and multiplicative in the number of models, i.e. $N_M \times N$, where N_M is the number of models. For QC algorithms the computational time complexity is additive on a $\log_2$ scale in N_M since $\log_2(N_M \times N) = \log_2 N_M + \log_2 N$.[9]

The HHL algorithm involves Hermitian matrices. The matrix $\mathcal{A}_N$ in equation (2.10) is not neces-

[9]This argument is based on the fact that one can stack models, such as $\{\bar{\nu}^m_{N_D}(\theta_m) : m = 1, \ldots, N_M\}$, in a large dimensional system. See e.g. Hansen (2007) for an example.

sarily Hermitian, but we can define:

$$A = \begin{pmatrix} 0 & \mathcal{A}_N \\ \overline{\mathcal{A}}_N & 0 \end{pmatrix} \tag{3.1}$$

where $\overline{\mathcal{A}}_N$ is the convex conjugate of $\mathcal{A}_N$ and we can solve the equation $Ax = b$ where

$$b = \begin{pmatrix} b_N \\ 0 \end{pmatrix} \qquad x = \begin{pmatrix} 0 \\ \bar{\nu}_N \end{pmatrix}$$

where $\bar{\nu}_N$ is the solution to the asset pricing equations appearing in equations (2.6) and (2.10) which is our main objective. Note also that equation (3.1) implies that A is of dimension $2 \times N$, with N the number of quadrature abscissa used in the discrete approximation formula. We suppress the dependence on N for A, b and x in order to simplify notation, although we will revisit the impact of N at a later stage. Moreover, in our applications the matrices $\mathcal{A}_N$ and vectors b_N have real-valued entries, such that we can replace Hermitian with symmetric and complex conjugate with transpose, but for the sake of generality we stick with the QC jargon and setup. That being said, when it comes to the actual implementation we will take advantage of the fact that there are no imaginary parts to the inputs. Moreover, we work with unit vectors, and therefore we scale the inputs accordingly.

We will denote the dimension of $|b\rangle$, the quantum state associated with b, as n_b which is also the dimension of the solution $|x\rangle$.[10] The HHL algorithm involves five main components, namely (1) state preparation - which prepares a quantum state that encodes the input vector b, (2) quantum phase estimation (QPE) to extract the eigenvalues of the matrix A, (3) ancilla bit rotation, (4) inverse quantum phase estimation (IQPE), and finally (5) measurement to extract (a function of) the solution vector x. More specifically, the first step of the algorithm is to write $|b\rangle$ in terms of the eigenvectors $|a_i\rangle$ corresponding to the eigenvalues λ_i of A, namely: $|b\rangle = \sum_{i=0}^{2^{n_b}-1} \beta_i |a_i\rangle$, we then find the solution as: $|x\rangle = \sum_{i=0}^{2^{n_b}-1} \lambda_i^{-1} \beta_i |a_i\rangle$.[11] So, the second step is finding the eigenvectors and eigenvalues of A. This is

[10] QC works with unit vectors, so whenever $\| |b\rangle \| \neq 1$, we normalize $|b\rangle$, i.e. replace it with $|b\rangle / \| |b\rangle \|$, to avoid overburdening the notation we keep using $|b\rangle$ as a unit vector.

[11] A typical QC algorithm relies on superpositions, representing a combination of multiple states simultaneously. $|b\rangle = \sum_{i=0}^{2^{n_b}-1} \beta_i |a_i\rangle$, is an example of superposition involving combinations of $|a_i\rangle$. In general the β_i can be complex-valued,

achieved by so-called Quantum Phase Estimation (QPE) which consists of controlled rotations applied to quibts and a Inverse Quantum Fourier Transform (IQFT). While this yields a solution, it is one with the eigenvectors of A as the basis, while we want the solution to be in terms of a standard orthonormal basis, i.e. $|u_i\rangle$. Therefore we need to undo the IQFT, or put differently applying QFT, which is next step in the algorithm. The final step is measurement, i.e. reading the output of the solution $|x\rangle$ we obtained. Quantum measurement is a fundamental concept in quantum mechanics that refers to the process of obtaining information about the properties of a quantum system. When a measurement is made, the system collapses into one of its possible states, known as an eigenstate, with a certain probability.

So far, we described $|x\rangle$ as the generic solution to a quantum linear algebra problem. At this point we need to remind ourselves that we are computing the solution to a dynamic asset pricing problem and that the inversion yields a quantum state best described as $|\bar{\nu}_N(\theta)\rangle$. To simplify notation from now on we will drop the N subscript, and write $|\bar{\nu}(\theta)\rangle$ instead of $|\bar{\nu}_N(\theta)\rangle$. The model solution encodes the equilibrium price-dividends sorted from low to high. Assuming that moments exist, one can in principle recover the equilibrium asset pricing moments from $|\bar{\nu}(\theta)\rangle$ for any given θ (cfr. Assumption 2.4).

We also introduce a quantum state pertaining to the data, which we write as $|d\rangle$. Assuming finite support for price-dividend ratios (an assumption also made by Tauchen and Hussey (1991)), we can divide up the interval between the empirical min and max of observed price-dividend ratios with the min as the first and the max as the N^{th} point and thus construct an (equally-spaced) empirical counterpart for the purpose of model comparisons with the data.[12] From this N-dimensional vector we can create the n_ν-dimensional quantum state $|d\rangle$. It is worth reminding ourselves at this point that all quantum states are unit-norm. This means that the construction of $|d\rangle$ also involves scaling to have a unit norm representation. Next, we impose the following technical condition:

ASSUMPTION 3.1. *For any model $|\bar{\nu}(\theta)\rangle$ the following holds: $|\bar{\nu}(\theta)\rangle \neq |d\rangle$.*

The above assumption implies that there is no perfect fit. At this stage we do not discuss the

an issue further discussed in the next section.

[12]Note that the equally spaced grid points may not coincide with the quadrature abscissa. Although we do not cover the case $N \to \infty$, it is reasonable to assume those differences will vanish with finer grid points.

reasons. It may be model specification error, but it may also be the result of approximation error in $|\bar{\nu}(\theta)\rangle$ and/or sampling error in $|d\rangle$. This will be covered in Section 7.

4 Decision-theoretic Foundations of QC Output Measurement

So far we have achieved the following: given a model of the type appearing in equation (1.1), we have a quantum state $|\bar{\nu}(\theta)\rangle$ which encodes the equilibrium asset pricing properties. In experimental physics one builds instruments to *measure* the quantum state. When the state of a quantum system is measured, it interacts with a classical measurement device, described mathematically by an operator. For QC applications in economics and finance, we are obviously not in a situation with lab measurement instruments, so we need to think about what to do once we obtain the solution $|\bar{\nu}(\theta)\rangle$. It is argued in this paper that we can take the mathematical framework of quantum computing to our advantage when we cast it as a decision-theoretic problem. We suggest to rely on decision-theoretic foundations in the spirit of Hansen and Sargent (2023) to capture quantities of interest pertaining to parameter uncertainty and ambiguity/doubt of the dynamic asset pricing models we want to study. To do so, we bring together elements from different strands of the literature. The purpose is not to be innovative in terms of theoretic foundations of decision making. The innovation is to bring together elements of (a) quantum computing and measurement, (b) statistical decision theory and (c) quantum models of decision-making.

The measurement process is probabilistic, meaning that the outcome of the measurement is not deterministic but instead determined by properties of the quantum state and the measurement operator. Harrow et al. (2009) suggest that when applying their algorithm, one should consider not looking at the actual solution $|x\rangle$, but rather compute as output $\langle x| \mathbb{A} |x\rangle$, in the spirit of the von Neumann model of measurement in quantum mechanics, for some judiciously chosen operator $\mathbb{A}$.

There is a small but growing literature in mathematical economics and mathematical psychology which tries to apply methods of quantum mechanics to decision making problems using the framework of a Hilbert space of wave functions. Aerts (2009), Aerts and Sozzo (2012) and Aerts et al. (2012) present a framework for modeling decision making under ambiguity by quantum mechanic methods

and apply it to solve the Ellsberg paradox.[13] Yukalov and Sornette (2010) and Yukalov and Sornette (2011) model decision making under uncertainty also by a Hilbert space of wave functions and study several paradoxes of decision theory, focusing in particular on the disjunction and conjunction effects.[14] Cerreia-Vioglio et al. (2018) use a notion of co-monotonicity for subspaces of self-adjoint operators to characterize a Choquet expected utility functional as preference representation. Literature reviews appear in Pothos and Busemeyer (2013) and Eichberger and Pirner (2018).

Recall the discussion in Section 2.2 where we noted that decision theories that build on representations of subjective and objective uncertainties articulated by Anscombe and Aumann (1963) involved a formal treatment of (a) states, (b) acts and (c) prizes/outcomes/consequences. We devote a subsection to each.

4.1 States

Table 4.1 has the same structure as the one appearing in the Introduction, which displayed the connections between the analysis in our paper with prior literature on (a) axiomatic foundations of decision making, (b) quantum behavioral models of decision making and (c) decision-theoretic foundations of statistical inference. As noted in the two right panels of Table 4.1, we do not exactly follow the setup of Hansen and Sargent (2023) when it comes to defining states, but instead are closer to the quantum behavioral framework appearing in the lower left corner of the table. An orthonormal basis $|u_i\rangle$, i = 0, . . ., $n_\nu - 1$ of tensor product space $\bigotimes_{i=1}^{n_\nu} \mathbb{H}$ represents the states and a model is a superposition of these basis states: $|\bar{\nu}(\theta)\rangle = \sum_{i=0}^{n_\nu - 1} \gamma_i^\nu |u_i\rangle$. Note that we directly work with a quantum state representation of the asset pricing model, instead of making the parameters the states. There are at least two reasons for this. First, the QC algorithms produce as output $|\bar{\nu}(\theta)\rangle$, and we are linking this output to quantum measurement. Second, the mapping from θ to $\bar{\nu}(\theta)$ is typically highly nonlinear and therefore not convenient to cast in a Hilbert space setting. Note that Assumption 2.4 implies there is a unique parameter vector associated with a solution $\bar{\nu}(\theta)$ and vice versa.

[13] Diederik Aerts and co-authors interpret the elements of a complex Hilbert space as determinants of probabilities and actions as projectors from the basic elements of the Hilbert space to outcomes or payoffs.

[14] Vyacheslav Yukalov and Didier Sornette interpret wave functions as "intentions" or "intended actions".

REMARK 4.1. *Since a model solution is a superposition of quantum states* $|\bar{\nu}(\theta)\rangle = \sum_{i=0}^{n_\nu - 1} \gamma_i^\nu |u_i\rangle$, *where* $|u_i\rangle$, $i = 0, \ldots, n_\nu - 1$ *an orthonormal basis of tensor product space* $\bigotimes_{i=1}^{n_\nu} \mathbb{H}$, *this means that (a) we look at states and acts that are more complex than the (convex hull involving real-valued mixing combinations of) Anscombe and Aumann (1963) states and acts and (b) we will work with von Neumann's (1932) quantum probability algebra instead of Kolmogorov's (1933) axiomatic foundations of probability (sometimes called classical) on which the decision theory used by Hansen and Sargent (2023) and the many papers cited by them, rely on. This will be further discussed in Section 5.*[15]

Next we distinguish pure from mixed quantum states formulated specifically for the purpose of our analysis.[16]

DEFINITION 4.1. *A pure state is represented by a single model* $\bar{\nu}(\theta)$. *A mixed state is a statistical mixture of multiple pure states (i.e. models) represented via density matrices (also known as density operators). A mixture of quantum states is again a quantum state. We will call mixed states of models Anscombe and Aumann (1963) lotteries (see Section 4.4 for further discussion).*[17]

Recall from the discussion in the previous section that instead of considering the pure states $|d\rangle$ and $|\bar{\nu}(\theta)\rangle$, econometricians and statisticians would want to look at $|d - \bar{\nu}(\theta)$. Note that the quantum state $|d - \bar{\nu}(\theta)\rangle$ is no longer unit norm. This will matter once we make comparisons across models, and will be further explored in Remark 4.3 and Definition 4.3.

4.2 Acts and Preferences for a Pure State

In Definition 4.1 we distinguished pure state, i.e. in our setting a single asset pricing model, from mixed states, a statistical ensemble of multiple models. Here we focus on pure states and thus a single model.

[15] More specifically, in the von Neumann quantum mechanics setting, probabilities are determined by the squared modulus of amplitudes (which are complex numbers) of superposition. This introduces a so called quantum interference term, which will be the source of quantum ambiguity.

[16] The notions of pure and mixed states is widely studied in quantum mechanics and information theory, see e.g. Nielsen and Chuang (2010). The definition provided here is tailored to the application in the current paper.

[17] More specifically a pure state is represented by a state vector $|\phi\rangle$ in the Hilbert space. The density matrix ρ for a pure state is represented by the projection operator $\rho = P_\phi = |\phi\rangle \langle\phi|$. The trace of the density matrix is equal to one. A mixed state is a statistical ensemble of pure states. Put differently, it is a combination using real-valued, positive probabilities of different states. It is described by a density matrix ρ that is a positive semi-definite, Hermitian operator with a trace equal to 1. It can be characterized as a convex combination of the density matrices corresponding to different pure states, $\rho = \sum_i p_i |\phi_i\rangle \langle\phi_i|$ where p_i are probabilities associated with the pure states.

Table 4.1: **States**: Left column based on Eichberger and Pirner (2018). Upper right based on Hansen and Sargent (2023). More references appear in Table 1.1.

Axiomatic Foundations of Decision Making	$\Rightarrow$	**Decision-theoretic Foundations of Statistical Inference**
States are observable		States can be latent
$\omega \in \Omega$ mutually exclusive		Parameters of statistical model are states
		with $(\Theta, \mathfrak{G})$ a measurable space of potential states
		$\mathscr{P}$ set of probability measures over states
		For each $p \in \mathscr{P}$, $(\Theta, \mathfrak{G}, p)$ is a probability space
$\Downarrow$		$\Downarrow$
Quantum Behavioral Models of Decision Making	$\Rightarrow$	**This paper**
States are latent		States are latent
States: orthonormal basis $\|\omega\rangle$ for $\omega = 0, \ldots, \|\Omega\| - 1$.		States: $\|u_i\rangle$, i = 0, …, $n_\nu - 1$ an orthonormal basis
of tensor product space $\mathbb{H}^{\|\Omega\|} \equiv \bigotimes_{i=1}^{\|\Omega\|} \mathbb{H}$		of tensor product space $\bigotimes_{i=1}^{n_\nu} \mathbb{H}$
		Model: $\|\bar{\nu}(\theta)\rangle = \sum_{i=0}^{n_\nu - 1} \gamma_i^\nu \|u_i\rangle$
		Data: $\|d\rangle = \sum_{i=0}^{n_\nu - 1} \gamma_i^d \|u_i\rangle$

Harrow et al. (2009) suggest upon solving a quantum matrix inversion with solution $|x\rangle$ to compute $\langle x| A |x\rangle$ for some judiciously chosen operator A. Aerts (2009), Yukalov and Sornette (2010), among others, consider an orthonormal basis $|\omega\rangle$ for $\omega = 0, \ldots, |\Omega| - 1$ as AA states and compute utility of an AA act represented by the unitary operator $\mathbb{A}$ as $\langle\omega| \mathbb{A} |\omega\rangle$. The operator $\mathbb{A}$ is sometimes called *state of mind* by Eichberger and Pirner (2018) among others. It therefore seems like we have a perfect alignment.

Continuing with what econometricians and statisticians typically look at, for any act (or measurement operator) $\mathbb{A}$, applied to the pure state $|d - \bar{\nu}(\theta)\rangle$ we compute expectation values using a well-known formula in quantum mechanics for the expectation value of an observable.[18] Namely we

[18] To be more precise, the Born rule for pure states $|\phi\rangle$ and a measurement operator $\mathbb{A}$ with eigenvectors $|a_i\rangle$ corresponding to eigenvalues λ_i (assuming for simplicity that they are unique) states that the probability distribution for the measurement outcomes $\mathbb{P}(\lambda_i) = |\langle a_i|\phi\rangle|^2$ and the expectation value is therefore $\sum_i \lambda_i |\langle a_i|\phi\rangle|^2 = \langle\phi| \mathbb{A} |\phi\rangle$, see e.g. Scherer (2019) p. 30. The Born rule was originally stated as a postulate and later proven for Hilbert spaces of dimension 3 or more as a theorem (known as the Gleason theorem). For a mixed state ρ (cfr. footnote 17) and measurement operator the expectation is $\text{tr}(\mathbb{A}\rho)$ and the Born rule states $\mathbb{P}(\lambda_i) = \text{tr}(P_i\rho)$. A special case of a mixed state is a pure one for which the expectation value is $\text{tr}(\mathbb{A}P_\phi)$. In equation (4.1) we use interchangeably the formulas $\langle d - \bar{\nu}(\theta)|\mathbb{A}|d - \bar{\nu}(\theta)\rangle$ and $\text{tr}(\mathbb{A}P_{d-\bar{\nu}(\theta)})$ in the case of pure states.

Table 4.2: **Acts**: Left column based on Eichberger and Pirner (2018). Upper right based on Hansen and Sargent (2023). More references appear in Table 1.1.

Axiomatic Foundations of Decision Making	$\Rightarrow$	**Decision-theoretic Foundations of Statistical Inference**
$a \in \mathcal{A}$		An act is a $\mathfrak{G}$-measurable function: $f : \Theta \to \Lambda$
$\mathcal{A} =: \{a : \Omega \to \mathcal{O}\}$		For given $\theta \in \Theta$, $f(\theta)$ is a lottery over prizes $x \in X$
$\Downarrow$		$\Downarrow$
Quantum Behavioral Models of Decision Making	$\Rightarrow$	**This paper**
State of mind represented by self-adjoint operator $\mathbb{A}$		Self-adjoint operator $\mathbb{A} : \bigotimes_{i=1}^{n_\nu} \mathbb{H} \to \bigotimes_{i=1}^{n_\nu} \mathbb{H}$
		See Table 4.3 for further discussion

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have the expectation value:

$$\langle d - \bar{\nu}(\theta)|\mathbb{A}|d - \bar{\nu}(\theta)\rangle = \text{tr}(\mathbb{A}P_{d-\bar{\nu}(\theta)}) \quad \text{with } P_{d-\bar{\nu}} = |d - \bar{\nu}(\theta)\rangle\langle d - \bar{\nu}(\theta)|, \tag{4.1}$$

where $P_{d-\bar{\nu}}$ is the projection operator on the space spanned by $|d - \bar{\nu}(\theta)\rangle$. Recall that both the data and model are exposed to the orthonormal basis states: $|u_i\rangle$, i = 0, . . ., $n_\nu - 1$ with $|\bar{\nu}(\theta)\rangle = \sum_{i=0}^{n_\nu-1} \gamma_i^\nu |u_i\rangle$ and $|d\rangle = \sum_{i=0}^{n_\nu-1} \gamma_i^d |u_i\rangle$. This means we can write the above equation as:

$$\langle d - \bar{\nu}(\theta)|\mathbb{A}|d - \bar{\nu}(\theta)\rangle = \sum_{i=0}^{n_\nu-1} |\gamma_i^\nu - \gamma_i^d|^2 \langle u_i| \mathbb{A} |u_i\rangle , \tag{4.2}$$

and if we define $p_j^\nu = \gamma_j^\nu - \gamma_j^d|^2 / \sum_{i=0}^{n_\nu-1} |\gamma_i^\nu - \gamma_i^d|^2$, we have the following expected loss formulation:

$$E[\mathcal{L}(\mathbb{A}, d - \bar{\nu}(\theta))] \equiv \sum_{i=0}^{n_\nu-1} p_i^\nu U(|u_i\rangle , \mathbb{A}) = \frac{1}{\sum_{i=0}^{n_\nu-1} |\gamma_i^\nu - \gamma_i^d|^2} \langle d - \bar{\nu}(\theta)|\mathbb{A}|d - \bar{\nu}(\theta)\rangle. \tag{4.3}$$

Note that if we define utility as the negative of loss, we can maintain the maximizing expected utility interpretation of the von Neumann-Morgenstern setting.[19] We will henceforth use loss instead of utility.

[19]This expectation can be computed via repeated applications of the quantum circuit, called *shots* in the jargon of QC, and averaging the outcomes. See Section 7 for further discussion.

REMARK 4.2. *It is beyond the scope of the current paper to digress on the axiomatic foundations for a von Neumann and Morgenstern (1944) expected utility representation of preference rankings among Anscombe and Aumann acts based on $\langle u_i|\mathbb{A}|u_i\rangle$ represented by self-adjoint operators $\mathbb{A}$ applied to AA states $|u_i\rangle$, i = 0, ..., $n_\nu - 1$, an orthonormal basis of tensor product space of tensor product space $\bigotimes_{i=1}^{n_\nu} \mathbb{H}$. However, see Table 1.1 and literature reviews appearing in Pothos and Busemeyer (2013) and Eichberger and Pirner (2018), among others, for further discussion.*

An interpretation more amenable to econometric analysis when $\mathbb{A} = P_d$, i.e. projection on the data quantum state $|d\rangle$, is the following (see equation A.1):

$$\begin{aligned}
\langle d - \bar{\nu}(\theta)|P_d|d - \bar{\nu}(\theta)\rangle &= \langle d - \bar{\nu}(\theta)|P_d P_d|d - \bar{\nu}(\theta)\rangle \quad P_d \text{ idempotent} \\
&= \langle e(\theta)\,|e(\theta)\rangle \qquad \text{where } |d - P_d\bar{\nu}(\theta)\rangle = |e(\theta)\rangle \\
&= \|\,|e(\theta)\rangle\,\|^2 \\
&= (1 - \text{Re}(\langle d\,|\bar{\nu}(\theta)\rangle))^2 + \text{Im}(\langle d\,|\bar{\nu}(\theta)\rangle)^2 \quad \text{see equation } (A.1).
\end{aligned} \tag{4.4}$$

We can interpret this as $\mathbb{A} = \mathbb{I}$ applied to the difference between the data quantum state $|d\rangle$ and the projection of the model quantum state $|\bar{\nu}(\theta)\rangle$ onto the space spanned by $|d\rangle$, i.e. $|P_d\bar{\nu}(\theta)\rangle$. Hence, we have an expected loss expressed as $\|\,|e(\theta)\rangle\,\|^2$ akin to the ℓ_2 distance between quantum states pertaining to data versus model filtered by the data. This is reminiscent of Cramer-von Mises statistics because $|d\rangle$ can be interpreted as a quantum state reflecting the empirical cumulative density and $|\bar{\nu}(\theta)\rangle$ the model counterpart. Alternatively, we can think of the expected loss in terms a measure of fit between zero (perfect fit with $\text{Re}(\langle d\,|\bar{\nu}(\theta)\rangle) = 1$ and the imaginary part is zero) and one (model and data are orthogonal).

REMARK 4.3. *There is, as equation (4.3) suggests, a scaling issue. Namely, while $|d\rangle$ and $\bar{\nu}(\theta)$ are unit norm, the state $|d - \bar{\nu}(\theta)\rangle$ isn't necessarily. As long as Assumption 3.1 holds, hence $\|d - \bar{\nu}(\theta)\| \equiv \sqrt{\langle d - \bar{\nu}(\theta)|d - \bar{\nu}(\theta)\rangle} \neq 0$, and we focus on a pure state (model), the (re)scaling does not matter for comparing acts $\mathbb{A}$, a point also highlighted by Basieva et al. (2018). Hence, for the moment we ignore scaling issues.*[20]

[20]Note also that the literature on quantum behavioral models of decision making does not cover the the issue of differently scaled states, only a single state which is not necessarily a unit norm vector.

4.3 Acts and Preferences with (Classical) Doubts about a Model

We now define a set of acts expressing doubts about a model. Recall from Remark 4.1 that we will consider von Neumann's (1932) quantum probability algebra instead of Kolmogorov's (1933) axiomatic foundations of probability and therefore look at acts that are more complex than those used by Anscombe and Aumann (1963) and Hansen and Sargent (2023), among others. For the moment we will only explore classical, i.e. based on Kolmogorov axiomatic foundations, acts (whereas in the next section we will explore quantum enhanced ambiguity/doubt acts). For this we define a set which is a combination of the model projection operator $P_{\bar{\nu}(\theta)} = |\bar{\nu}(\theta)\rangle \langle \bar{\nu}(\theta)|$ and some act $\tilde{\mathbb{A}}$ which is not orthogonal to the model error quantum state $|d - \bar{\nu}(\theta)\rangle$, namely:

DEFINITION 4.2 (Classical Ambiguity/Doubt AA Acts)**.** *Let the self-adjoint operator* $\tilde{\mathbb{A}} : \bigotimes_{i=1}^{n_\nu} \mathbb{H} \rightarrow \bigotimes_{i=1}^{n_\nu} \mathbb{H}$ *be such that* $\tilde{\mathbb{A}} \neq P_{|\bar{\nu}(\theta)\rangle}$, $\tilde{\mathbb{A}} \neq P_{|d\rangle}$ *and tr*$(\tilde{\mathbb{A}} P_{d-\bar{\nu}(\theta)}) \neq 0$*. Then for* $0 \leq p < 1$ *define the sets:*

$$\mathcal{A}^{CD}_{|\bar{\nu}(\theta)\rangle}(p) = \{\mathbb{A} : \mathbb{A} = (1-p)P_{\bar{\nu}(\theta)} + p\tilde{\mathbb{A}}\}. \tag{4.5}$$

Note that the set in the above definition with $p = 0$ represents the act of no doubts, or full confidence in the model, namely $\mathcal{A}^{CD}_{|\bar{\nu}(\theta)\rangle}(0)$ contains the single element $P_{|\bar{\nu}(\theta)\rangle}$. We use the superscript CD to refer to classical doubt.

REMARK 4.4. *It is worth emphasizing at this point that the ambiguity/doubts about models are those of the econometrician/statistician, not the agents in the model whose attitudes towards risk and uncertainty are embedded in the functions* $\lambda(y, x)$*,* $f(y|x)$*, and* $\omega(y)$*, in equation (2.2). This is in contrast to the models in the quantum decision-theoretic literature designed to describe agent's attitudes towards risk and uncertainty. By the same token, while the asset pricing problem is dynamic, the decision problem considered by the econometrician is static. Dynamic extensions involve complexities regarding updating of beliefs about model specification are beyond the scope of the current paper.*

The following is labeled the *No doubts* theorem, telling us that without ambiguity about the model being considered, i.e. $|\bar{\nu}(\theta)\rangle$ is being acted on with full confidence, then (and only then) purely data driven acts P_d and purely model driven ones are equivalent.

THEOREM 4.1 (No doubts)**.** *Consider the pure states (a)* $|d - \bar{\nu}(\theta)\rangle$*, and (b) acts* $\mathbb{A} \in \mathcal{A}^{CD}_{|\bar{\nu}(\theta)\rangle}(p)$*,* $0 \leq p < 1$*. Then* $E[U(P_d, d - \bar{\nu}(\theta))] = E[U(\mathbb{A}, d - \bar{\nu}(\theta))]$ *if and only if* $\mathbb{A} = P_{\bar{\nu}}$*.*

Proof: see Appendix A.1.

Once there is doubt about the model being examined, purely data-driven acts may be better or worse. Namely, whereas $\text{tr}(\tilde{\mathbb{A}}P_{d-\bar{\nu}(\theta)}) \neq 0$, it may be either positive or negative and for $\mathbb{A} \in \mathcal{A}^{CD}_{|\bar{\nu}(\theta)\rangle}(p)$: $\text{tr}(\tilde{\mathbb{A}}P_{d-\bar{\nu}(\theta)}) = (1-p)\text{tr}(P_{|\bar{\nu}(\theta)\rangle}P_{d-\bar{\nu}(\theta)}) + p\text{tr}(\tilde{\mathbb{A}}P_{d-\bar{\nu}(\theta)})$, as shown in the proof of Theorem 4.1. The following corollary will be useful when we discuss quantum extensions of ambiguity/doubts:

Corollary 4.1. *Consider the pure state $|d-\bar{\nu}(\theta)\rangle$, combined with act $\mathbb{A} = (1-p)P_{\bar{\nu}} + pP_d$. Then $E[U(P_d, d-\bar{\nu}(\theta))] = E[U(\mathbb{A}, d-\bar{\nu}(\theta))] = E[U(P_{\bar{\nu}}, d-\bar{\nu}(\theta))]$.*

The above corollary tells us that classical doubts about the data versus model don't matter in terms of expectations. The result is obtained from the linearity of the trace and the result in Theorem 4.1.

The statistical act brings us back to the scaling issue regarding $|d-\bar{\nu}(\theta)\rangle$. For the purpose of comparison across different models (i.e. not different acts for a given model discussed so far), we introduce a *benchmark* model, denoted by $\bar{\nu}_B$, which may or may not depend on parameters (and in the latter case we suppress them). In many applications we have a strawman model which is often parameter-free or is based on a simple model such as constant or AR(1) argument. We will denote by s_B the scale of $\langle d-\bar{\nu}_B|d-\bar{\nu}_B\rangle$.[21]

DEFINITION 4.3. *For the purpose of comparing acts $\mathbb{A}$ across models $\bar{\nu}(\theta)$ we define the following expected loss measure:*

$$E[\tilde{\mathcal{L}}_B(\mathbb{A}, d-\bar{\nu}(\theta))] \equiv \frac{s_B}{s_{\bar{\nu}(\theta)}} tr(\mathbb{A}P_{d-\bar{\nu}(\theta)}) \quad \textit{with} \quad s_{\bar{\nu}(\theta)} = \langle d-\bar{\nu}(\theta)|d-\bar{\nu}(\theta)\rangle \tag{4.6}$$

with respect to benchmark $s_B = \langle d-\bar{\nu}_B|d-\bar{\nu}_B\rangle$ used across all models.

The expectation value in (4.6) is scaled across all models and acts vis-à-vis the benchmark.[22] Note that models with $s_B/s_{\bar{\nu}(\theta)} > 1$ provide a boost in expectation value. This means that models with a "statistical" fit better than the benchmark model are favored. One word of caution though, is that any particular act $\mathbb{A}$ may undo such advantage, notably with possible doubts for whatever reason, about the model. With N_M plausible models, each associated with pure quantum states $|\bar{\nu}^m(\theta_m)\rangle$ for $m = 1, \ldots, N_M$ we can now compare $(s_B/s_{\bar{\nu}^m(\theta_m)})\text{tr}(P_{d-\bar{\nu}^m(\theta_m)}\mathbb{A})$ across models for a given AA act $\mathbb{A}$.

[21] It is assumed that Assumption 3.1 applies to the benchmark model, and therefore $a_B \neq 0$.

[22] The formula is for a so called pure state, which we will extend to mixed states when we will discuss lotteries.

We conclude with highlighting two AA acts we will use in the next section. The first is:

$$\begin{aligned} \mathbb{A} = (1-p)P_{\bar{\nu}(\theta)} + pP_B \qquad & \tilde{\mathbb{A}} = P_B = |\bar{\nu}_B\rangle \langle \bar{\nu}_B| \\ E[\tilde{\mathcal{L}}_B(((1-p)P_{\bar{\nu}(\theta)} + pP_B, d - \bar{\nu}(\theta))] &= (1-p)\text{tr}(P_{\bar{\nu}(\theta)}P_{d-\bar{\nu}(\theta)}) + p\text{tr}(P_B P_{d-\bar{\nu}(\theta)}), \end{aligned} \tag{4.7}$$

reflecting ambiguity of a model $\bar{\nu}(\theta)$ versus the benchmark one. The second is:

$$\begin{aligned} \mathbb{A} = (1-p)P_d + pP_B \qquad & \tilde{\mathbb{A}} = P_B + \frac{p-1}{p}[P_d - P_{\bar{\nu}(\theta)}] \\ E[\tilde{\mathcal{L}}_B((1-p)P_d + pP_B, d - \bar{\nu}(\theta))] &= (1-p)\text{tr}(P_d P_{d-\bar{\nu}(\theta)}) + p\text{tr}(P_B P_{d-\bar{\nu}(\theta)}), \end{aligned} \tag{4.8}$$

where one has doubts about the data and puts theory, via the benchmark model, ahead of measurement as p moves closer to one.

4.4 Lotteries and Horse Races

We now turn to lotteries of models, i.e. mixed quantum states which are statistical mixture of models. These mixture reflect uncertainty, as opposed to ambiguity/doubts. In Table 4.3 we look at the four possible cases: No/Yes Classical Ambiguity and No/Yes Uncertainty. So far we covered the 'No Uncertainty' top two entries, i.e. with or without (classical) ambiguity. Recall that we considered N_M models: $\bar{\nu}_1(\theta_1)$, ..., $\bar{\nu}_{N_M}(\theta_{N_M})$. Suppose now that we have some objective probability distribution across these models, attaching probability p_j to model j, with $\sum_j p_j = 1$. This could come from an asymptotic sampling argument for instance as we will pursue in the example presented in the implementation Section 6. In quantum mechanics parlance we construct the mixed state, which in Anscombe and Aumann (1963) parlance is a lottery of models $Q = \sum_j p_j P_{\bar{\nu}_j}$, $\sum_j p_j = 1$ and act on it with $\mathbb{A}$. In Table 4.3 we set $\mathbb{A} = P_d$.The ensuing generic formula is again in equation (4.6) for the expectation value, this time for a lottery of models against the data. This is covered in the lower left quandrant of Table 4.3. The lower right panel refers to a situation alluded to in Remark 2.7 were we split the parameter vector $\theta \in \Theta$, with $\theta \equiv (\theta_L, \theta_H) \in \Theta_L \times \Theta_H$ with the former governed by some objective probability distribution, whereas the latter by a subjective one. Say we entertain two different SDFs, but agree on a lottery for state dynamics. In such a case the mind set is ambiguous about the SDF part

Table 4.3: **Lotteries and Horse races**

	Classical Ambiguity/Doubts: No	Classical Ambiguity/Doubts: Yes
No Uncertainty	Pure States: $\|d - \bar{\nu}(\theta)\rangle$ Acts: $\mathbb{A} = P_\nu$ or P_d	Pure States: $\|d - \bar{\nu}(\theta)\rangle$ Acts: $\mathbb{A} \in \mathcal{A}^{CD}_{\|\bar{\nu}(\theta)\rangle}(p)$, $0 < p < 1$
Yes	Mixed States (AA Lottery) $\|q\rangle = \sum_j p_j \|d - \bar{\nu}_j\rangle$ and $Q = \sum_j p_j P_{\bar{\nu}_j}$ $0 \leq p_j \leq 1 \; \forall \; j$ and $\sum_j p_j = 1$ Acts: $\mathbb{A} = P_d$ (for example) Exp. value: $\frac{s_B}{s_q} \sum_j p_j \text{tr}(P_d P_{\bar{\nu}_j})$	Mixed States $Q^1 = \sum_j p_j^1 P_{\bar{\nu}_j^1}$ and $\|q^1\rangle = \sum_j p_j^1 \left\|d - \bar{\nu}_j^1\right\rangle$ $0 \leq p_j^k \leq 1 \; \forall \; j$, $k = 1, 2 \quad Q^2 = \sum_j p_j^2 P_{\bar{\nu}_j^2}$ Acts: $\mathbb{A} = (1-p)Q^1 + pQ^2$ (for example) Exp. value: $\frac{s_B}{s_{q_1}} \text{tr}(\mathbb{A} Q^1)$

of the model reflected by $\mathbb{A}$ = (1-p)Q^1 + pQ^2, where Q^1 is a lottery of models with one SDF and Q^2 the lottery of models with the alternative SDF specification.

5 Beyond Classical Ambiguity

The potential of exponential speed-ups is one thing. Arguably of more importance is the fact that quantum computers allow us to entertain and implement notions of ambiguity beyond those primarily used in the econometrics literature. In Remark 4.1 we observed that a superposition of quantum states allows us look at states and acts that are more complex than the Anscombe and Aumann (1963) states and acts.

So far we have not fully exploited notions of superposition. We will rely on a simple pedagogical example at first, involving a superposition of the basis states $|0\rangle$ and $|1\rangle$, to introduce the topic. Namely, consider the following quantum state:

$$|\Psi\rangle = c_0|0\rangle + c_1|1\rangle$$

where both c_0 and $c_1 \in \mathbb{C}$, called amplitudes, with $|c_1|^2$ + $|c_1|^2$ = 1. The fact we allow for complex-

valued combinations is crucial. Take the following four examples:

$$|\Psi_1\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle \quad |\Psi_2\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{-1}{\sqrt{2}}|1\rangle$$

$$|\Psi_3\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{i}{\sqrt{2}}|1\rangle \quad |\Psi_4\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{-i}{\sqrt{2}}|1\rangle .$$

If we interpret the square of the moduli of the amplitudes, i.e. $|c_1|^2$ and $|c_1|^2$, as classical probabilities, then each imply a 50-50 chance coin toss with outcomes $|0\rangle$ and $|1\rangle$. But classical probability arguments do not apply here. Clearly, $|\Psi_1\rangle$ through $|\Psi_4\rangle$ are different quantum states, and the best way to highlight this is through a measurement operator $\mathbb{A}$ with eigenvectors $|a_i\rangle$ corresponding to eigenvalues λ_i applied to $|\Psi_j\rangle$, j = 1, . . ., 4. Applying the Born rule (cfr. footnote 18) the probability distribution for the measurement outcomes of $\mathbb{A}$ acting on state $|\Psi_j\rangle$ is $\mathbb{P}(\lambda_i) = |\langle a_i|\Psi_j\rangle|^2$, which differs across the states $|\Psi_1\rangle$ through $|\Psi_4\rangle$.[23]

Inspired by Eichberger and Pirner (2018), who use the same scheme to provide quantum decision-theoretic solutions to the Ellsberg (1961) paradox, we define quantum ambiguity based on the following superposition of quantum states:

DEFINITION 5.1 (Quantum Ambiguity/Doubt AA Acts)**.** *Consider two quantum states $|\mathcal{S}_1\rangle$ and $|\mathcal{S}_2\rangle$, with $0 \leq \alpha \leq 1$, and $\delta \neq 0$. We define the quantum ambiguity state:*

$$|\mathcal{S}_{1,2}(\alpha, \delta)\rangle = \alpha\,|\mathcal{S}_1\rangle + e^{i\delta}\sqrt{1-\alpha^2}\,|\mathcal{S}_2\rangle \tag{5.9}$$

and associated AA acts through the set of self-adjoint projection operators:

$$\mathcal{A}^{QD}_{|\mathcal{S}_{1,2}\rangle}(\alpha) = \{\mathbb{A} : \mathbb{A} = P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle} = |\mathcal{S}_{1,2}(\alpha,\delta)\rangle\,\langle\mathcal{S}_{1,2}(\alpha,\delta)|\,, \delta \neq 0\}. \tag{5.10}$$

with the QD subscript referring to quantum doubts, in contrast to $\mathcal{A}^{CD}_{|\bar{\nu}(\theta)\rangle}(p)$ in Definition 4.2.

Eichberger and Pirner (2018) emphasize in their analysis the important role to solving Ellsberg's paradox played by the complex-valued interference term $e^{i\delta}$. It plays an important role in our analysis as well, as expressed in the following theorem.

[23] More detailed discussions of quantum probability theory appear in, among others, Holevo (1982), Parthasarathy (2012) and Sakurai and Napolitano (2020).

Theorem 5.1 (Classical/Quantum Ambiguity Decomposition)**.** *Consider a model (pure state)* $|d - \bar{\nu}(\theta)\rangle$ *or mixture of model models (AA lottery)* $q = \sum_j p_j |d - \bar{\nu}_j\rangle$*. We will use the generic notation* $|\mathcal{S}_3\rangle$ *to represent the quantum state in either case. In addition, consider two quantum states* $|\mathcal{S}_1\rangle$ *and* $|\mathcal{S}_2\rangle$*, with* $0 \leq \alpha \leq 1$*, and* $\delta \neq 0$ *specified in Definition 5.1. Then the expectation value of measuring* $|\mathcal{S}_3\rangle$ *with projection operator* $P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle}$ *appearing equation (5.10) equals:*

$$tr(P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle}P_3) = \underbrace{\alpha^2 tr\,(P_1P_3) + (1-\alpha^2)tr\,(P_2P_3)}_{\textit{Classical Ambiguity}} + \underbrace{(\alpha\sqrt{1-\alpha^2})r_3^1 r_3^2 \cos(\delta + \alpha_3^{1.2})}_{\textit{Quantum Ambiguity}} \tag{5.11}$$

where $r_3^1 e^{i\alpha_3^1} = \langle \mathcal{S}_3|\mathcal{S}_1\rangle$*,* $r_3^1 e^{i\alpha_3^2} = \langle \mathcal{S}_3|\mathcal{S}_2\rangle$*, and* $\alpha_3^{1,2} = \alpha_3^2 - \alpha_3^1$*. Following Definition 4.3, comparing acts* $P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle}$ *across models* $|\mathcal{S}_3\rangle$ *we define the following expected loss measure:*

$$E[\tilde{\mathcal{L}}_B(P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle}, |\mathcal{S}_3\rangle)] \equiv \frac{s_B}{s_{|\mathcal{S}_3\rangle}} tr(P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle}P_3) \tag{5.12}$$

with $s_{|\mathcal{S}_3\rangle}$ *and* s_B *as defined in equation (4.6).*

Proof see Appendix Section A.2.

We learn from equation (5.11) that tr($P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle}P_3$) has two components. One reflects classical ambiguity with probability mix $p = \alpha^2$, and the other reflecting quantum ambiguity. The former is reminiscent of for example the two cases appearing in equations (4.7) and (4.8). The quantum ambiguity component, equal to $(\alpha\sqrt{1-\alpha^2})r_3^1 r_3^2 \cos(\delta + \alpha_3^{1.2})$ takes on values between $(\alpha\sqrt{1-\alpha^2})r_3^1 r_3^2$ for $\delta + \alpha_3^{1.2} = 0$, and $-(\alpha\sqrt{1-\alpha^2})r_3^1 r_3^2$ for $\delta + \alpha_3^{1.2} = \pi$. Hence, for a given $p = \alpha^2$ we have a range of quantum ambiguities determined by δ.

REMARK 5.1. *The paramters* α *and* δ *are chosen by the decision maker, i.e. the econometrician, and are not part of the model specification (recall also Remark 4.4). In the context of the Ellsberg paradox Oechssler and Roomets (2015) provide experimental evidence of ambiguity aversion on the part of the decision makers. Alternatively, we can also think of picking* α *only and look at all the possible* δ*s. This leads to the following definition.*

DEFINITION 5.2 (Quantum Ambiguity Ranking)**.** *For a given* $0 \leq \alpha \leq 1$*, a model or lottery of models represented by quantum state* $|\mathcal{S}_3\rangle$ *when measured with quantum ambiguity operator* $P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle}$ *is quantum unambiguously better than another model or lottery of models represented by quantum*

state $|\mathcal{S}_4\rangle$ with associated projection operator P_4 if (in terms of utility):

$$\max_{\delta} \frac{s_B}{s_{|\mathcal{S}_3\rangle}} tr(P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle} P_3) \leq \min_{\delta} \frac{s_B}{s_{|\mathcal{S}_4\rangle}} tr(P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle} P_4) \tag{5.13}$$

Next we turn to the two cases considered in equations (4.7) and (4.8).

COROLLARY 5.1. *Consider two AA acts considered in equations (4.7) and (4.8). With quantum ambiguity, we have for* $|\mathcal{S}_1\rangle = |\bar{\nu}(\theta)\rangle$, $|\mathcal{S}_2\rangle = |\bar{\nu}_B\rangle$, $|\mathcal{S}_3\rangle = |d - \bar{\nu}(\theta)\rangle$:

$$\begin{aligned} E[\tilde{\mathcal{L}}_B(P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle}, |d - \bar{\nu}(\theta)\rangle)] &= \underbrace{E[\tilde{\mathcal{L}}_B(((1-p)P_{\bar{\nu}(\theta)} + pP_B, d - \bar{\nu}(\theta))]}_{\text{Classical Ambiguity as in eq. (4.7)}} \\ &\quad + (\alpha\sqrt{1-\alpha^2}) r_3^1 r_3^2 \cos(\delta + \alpha_3^{1.2}) \end{aligned} \tag{5.14}$$

where $r_3^1 e^{i\alpha_3^1} = \langle \mathcal{S}_3 | \mathcal{S}_1 \rangle$, $r_3^1 e^{i\alpha_3^1} = \langle \mathcal{S}_3 | \mathcal{S}_2 \rangle$, *and* $\alpha_3^{1,2} = \alpha_3^2 - \alpha_3^1$. *and for* $|\mathcal{S}_1\rangle = |d\rangle$, $|\mathcal{S}_2\rangle = |\bar{\nu}_B\rangle$, $|\mathcal{S}_3\rangle = |d - \bar{\nu}(\theta)\rangle$:

$$\begin{aligned} E[\tilde{\mathcal{L}}_B(P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle}, |d - \bar{\nu}(\theta)\rangle)] &= \underbrace{E[\tilde{\mathcal{L}}_B((1-p)P_d + pP_B, d - \bar{\nu}(\theta))]}_{\text{Classical Ambiguity as in eq. (4.8)}} \\ &\quad + (\alpha\sqrt{1-\alpha^2}) r_3^1 r_3^2 \cos(\delta + \alpha_3^{1.2}). \end{aligned} \tag{5.15}$$

Regarding the second case, we will henceforth refer to $|\mathcal{S}_1\rangle = |d\rangle$ as the data quantum state, $|\mathcal{S}_2\rangle = |\bar{\nu}_B\rangle$ as the benchmark one and finally $|\mathcal{S}_3\rangle = |d - \bar{\nu}(\theta)\rangle$ the target model pricing error quantum state.

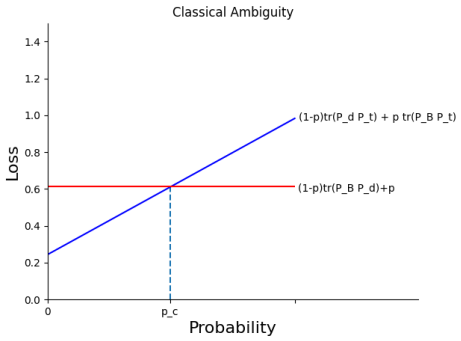
Next we revisiting a different interpretation of the formulas in this section. Namely, using arguments similar to those in equation (4.4), we can write:

$$\begin{aligned} \text{tr}(P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle} P_{d-\bar{\nu}(\theta)}) &= \langle d - \bar{\nu}(\theta) | P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle} | d - \bar{\nu}(\theta)\rangle \\ &= \langle e(\theta) \, | e(\theta)\rangle \qquad \text{where } \left| P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle} d - P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle} \bar{\nu}(\theta) \right\rangle = |e(\theta)\rangle \\ &= \| \, |e(\theta)\rangle \, \|^2 \end{aligned} \tag{5.16}$$

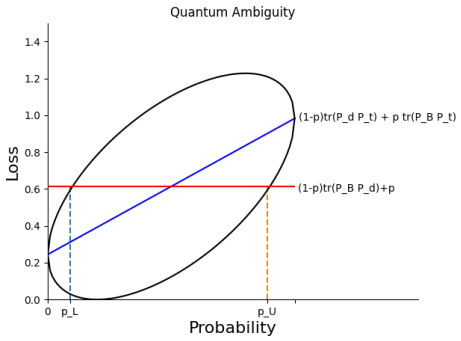
Therefore, we look at the model and the data quantum states with the *state of mind* $P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle}$, using terminology from the quantum decision theory literature. Note that in equation (5.16) we again have a ℓ_2 measure of discrepancy, but this time applied to both modified data and model quantum states.

In the remainder of this section and the next we will focus on $|\mathcal{S}_1\rangle = |d\rangle$, $|\mathcal{S}_2\rangle = |\bar{\nu}_B\rangle$ and $|\mathcal{S}_3\rangle = |d - \bar{\nu}(\theta)\rangle$, and will label the quantum states (a) data and associated projection operator P_d, (b) benchmark model and P_B and (c) the target model pricing error quantum state which we will associate with projection operator P_T.

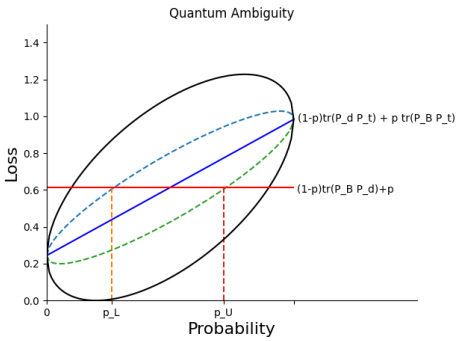
Figure 5.1: Classical versus Quantum Ambiguity



(a) Classical Ambiguity Only
$p_C = 0.49$



(b) Quantum Ambiguity I
$p_L = 0.09 \quad p_U = 0.90$



(c) Quantum Ambiguity II
$p_L = 0.26 \quad p_U = 0.72$

We proceed with a visualization of the setup in equation (5.15) appearing in Figure 5.1. The first panel (a) in the figure shows the classical ambiguity as a function of the probability, namely $p = \alpha^2$. The horizontal line corresponds to $0.5\text{tr}(P_d P_B) + 0.5\ \text{tr}(P_B P_B) = 0.5 + 0.5\text{tr}(P_d P_B)$. Hence, we take a 50-50 view of the benchmark model and the data. The upward sloping line represents the target model

pricing error against the classical mixture measurement operator. Hence, the linear line represents: $(1-p)$tr$(P_T P_d)$ + p tr$(P_T P_B)$. The idea is that we start with the 50-50 classical mix (we could start with something else as discussed later) and now we take a new model into account and assess it vis-à-vis the data and the benchmark model. It is important to recall Remark 4.4 noting that the decision problem considered by the econometrician is static. So, we do not consider each of the plots in Figure 5.1 as a sequence of decisions, but rather stand alone decisions.

We start with the classical ambiguity measurement operator with a mixture p. Above the point p_c we opt for the benchmark model which has a lower expected loss and left of p_C we opt for the target model. What that means is that with a classical measurement operator to look at the target model with a lot doubts about the benchmark model, i.e. p close to zero, meaning a lot of confidence in the data to measure the target model, we opt for the target model. Conversely, with strong doubts about the data we stick to the benchmark model in this case. The intersection point p_C = 0.49. Not much different from the 50-50 mix.

Panel (b) is the first of two quantum ambiguity scenarios. Things get more interesting here. The oval-shaped plot represents the extremes of the ambiguity component $(\alpha\sqrt{1-\alpha^2})r_3^1 r_3^2 \cos(\delta + \alpha_3^{1.2})$. Those extremes are obtained for $\cos(\delta + \alpha_3^{1.2})$ = 1 (and therefore $(\delta + \alpha_3^{1.2})$ = 0) and $\cos(\delta + \alpha_3^{1.2})$ = -1 (and therefore $(\delta + \alpha_3^{1.2}) = \pi$). Hence, the range is $[-(\alpha\sqrt{1-\alpha^2})r_3^1 r_3^2, (\alpha\sqrt{1-\alpha^2})r_3^1 r_3^2]$. Now, we have p_L and p_U. Below the former we pick for sure the target model, and above p_U we select for sure the benchmark model regardless of the values of p and δ. Note that r_3^1, r_3^2 and $\alpha_3^{1,2} = \alpha_3^2 - \alpha_3^1$ are determined by the choice of the benchmark model, the target model and the features of the data. We call the interval $[p_L, p_U]$ as inconclusive. By that we mean that unless we pick specific parameters p and δ, we don't have a quantum unambiguous answer between keeping the benchmark model or the new target model when measuring the quantum state of the latter with a quantum ambiguous measurement operator involving the data and benchmark model.

In panel (c) of Figure 5.1 shows a reduced oval-shaped region of quantum ambiguity and by implication a smaller inconclusive interval. What might be source of such shrinkage? First, we might consider a small range for δ. In the figure we restricted δ to vary between $3\pi/8$ and $5\pi/8$. That would be an example of the econometrician/decision maker committing to a narrower notion of quantum ambiguity. But equally important is the role of $\alpha_3^{1,2} = \alpha_3^2 - \alpha_3^1$, i.e. the angle between the data and the

benchmark in this example versus the target model and the benchmark model. Therefore, changing the benchmark and/or target model change the shape of the ambiguity area.

6 Illustrative Empirical Model Implementation

Let us first recap what we have achieved so far. Quantum states are either pure or mixed. Pure states represent a single model solution, whereas mixed states embed uncertainty about some features of the model. Measurement operators encode ambiguity about the data and a benchmark model. In this empirical illustrative example we showcase the various attributes of the procedure.

We start with equation (2) of Hansen et al. (2008) applied to quarterly log dividend growth.[24] Namely let $d_t = \log D_t$, and:

$$\begin{aligned} d_{t+1} - d_t &= a + x_{t+1} + be_{t+1} \\ x_{t+1} &= \rho_1 x_t + c\epsilon_{t+1} \end{aligned} \tag{6.1}$$

We collect the parameters of the above model into the vector $\theta_L = (a, b, c, \rho_1)'$. Assuming standard normal errors, we can estimate the parameters of the model (6.1) via Maximum Likelihood.[25] We rely on the asymptotic distribution theory for MLE to characterize model uncertainty about equation (6.1). Namely, we assume that the model is correctly specified, but we are uncertain about its parameter values. Hence, we don't question neither the functional form nor the distributional assumptions. This is done for the sake of simplicity and it reflects the notion of AA lotteries displayed in the lower part of Table 4.3. This will create mixed quantum states, and therefore reflect uncertainty. Namely, we create one thousand draws of parameter models $\{\theta_i, i = 1, \dots 1000\}$ and construct a 1000 models for log dividend growth. More specifically , we draw one thousand $\theta_L^i \sim N(\hat{\theta}_L, \hat{\Sigma})$, excluding draws with $\rho_1 < 0$ and $\rho_1 > 1$. The collection of models therefore represents statistical uncertainty, since we assume the model specification is correct and asymptotic distribution theory provides us guidance about parameter uncertainty. We can characterize and visualize the uncertainty using the Kullback-

[24] Hansen et al. (2008) focus on consumption growth and examine the risk pricing of future cash flows. For our purpose, it will be convenient to start with an equation linking dividend growth directly to a state variable process.

[25] Details about the estimation appear in Appendix Section A.3. The parameter estimates appear in Table A.1.

Leibler divergence $KL_i = \frac{1}{2}[(\theta_L^i - \hat{\theta}_L)'\hat{\Sigma}^{-1}(\theta_L^i - \hat{\theta}_L))]$. The fitted distribution appears in Appendix Figure A.2. This statistical ensemble is the basis to construct a mixed state benchmark model.

The log of the one period stochastic discount factor $s_{t+1,t}$, again following the setup of Hansen et al. (2008), is as follows (with parameter estimates discussed in Appendix Section A.3):

$$s_{t+1,t} = -0.8974 + 1.2038x_t + \xi w_{t+1}. \tag{6.2}$$

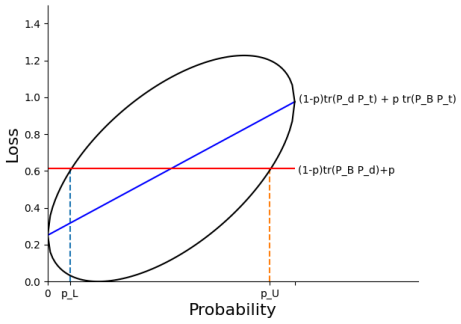
We focus only on the parameter ξ, and will consider two cases, determined by assumptions regarding the preferences of the representative agent. Namely, under CRRA preferences $\xi = -0.03630\gamma$, using the parameter estimates reported in Appendix Table A.1, with γ the coefficient of relative risk aversion and the slope is the estimate of the parameter pertaining to the dividend shock in equation (6.1). The second case involves recursive utility with the intertemporal elasticity of substitution equal to one, where $\xi = 0.0412 - 0.0775\gamma$, using parameter estimates reported in Appendix Section A.3.

In Appendix Section A.3 we describe the computations for a single model. All calculations use a four by four transition matrix, $N = 4$, which results in a 16-dimensional quantum linear system problem to create the state $|d - \bar{\nu}(\theta)\rangle$. We classically prepare the quantum state vector and calculate the expectation value of the observable using $Qiskit$.[26] As noted earlier, our classically computed expectation value is equivalent to the result of a hypothetical HHL implementation without hardware noise nor algorithmic error. Such an implementation would require a large number of evaluation qubits for QPE and a noiseless processor. Needless to say, our classical calculations do not pose an exponential speedup. However, they serve as a proof of concept for quantum implementations that will provide an exponential speedup once hardware is up to the task.

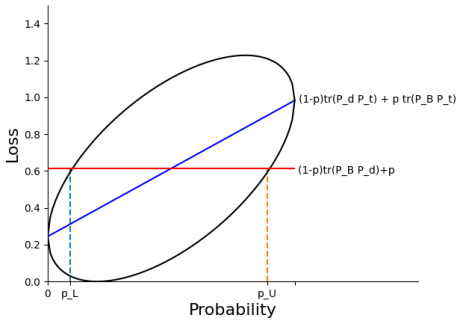
We fix $\gamma = 10$ and define the benchmark model as the CRRA specification with $|\bar{\nu}_B\rangle = \sum_j p_j |d - \bar{\nu}_j\rangle$ using the mixed state weights obtained from the Kullback-Leibler divergence, the weights are $p_j = KL_j / \sum_j KL_j$. A few words regarding doubt about the data quantum state $|d\rangle$. Recall that the functional form in equation (6.1) is given and the parameters are uncertain. Nevertheless, one may still argue that the dividend growth data is a source of concern and entertain the thought that observed

[26]Qiskit is an open-source software development kit for programming quantum computers. It provides tools to create, manipulate, and run quantum programs on IBM quantum devices. It is also widely used on other quantum hardware platforms and is the leading Python software development kit for open source quantum development.

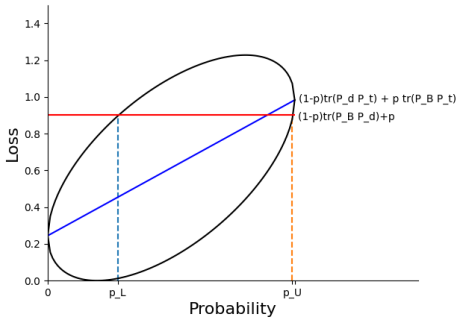
Figure 6.1: Illustrative Empirical Models



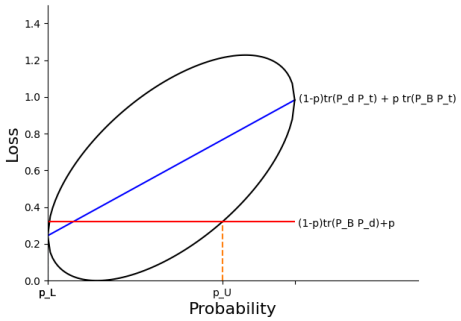
(a) Benchmark CRRA with p = 0.5
Target IES = 1, γ = 2
p_L = 0.09, p_C = 0.49, p_U =0.90



(b) Benchmark CRRA with p = 0.5
Target IES = 1, γ = 10
p_L = 0.09, p_C = 0.49, p_U =0.89



(c) Benchmark CRRA with p = 0.9
Target IES = 1, γ = 10
p_L = 0.28, p_C = 0.89, p_U =0.99



(d) Benchmark CRRA with p = 0.1
Target IES = 1, γ = 10
p_L = 0.00, p_C = 0.10, p_U =0.71

dividends are only a noisy proxy for economy-wide asset payoffs. This concern may be embedded in the measurement operator as doubt about the data.

Using the benchmark model we evaluate the IES models with γ = 2, 5 and 10 as target models. Hence, in our empirical illustration we have $|\mathcal{S}_1\rangle = |d\rangle$, $|\mathcal{S}_2\rangle = |\bar{\nu}_B\rangle$ the aforementioned CRRA model with γ = 10 and the mixed state statistical ensemble pertaining to dividend parameter uncertainty. Finally, the targets $|\mathcal{S}_3\rangle = |d - \bar{\nu}(\theta)\rangle$ are the three long-run risk models with the same uncertainty

regarding the parameter uncertainty.

As in the previous section, we look at a collection of static decisions (cfr. Remark 4.4) and the reference horizontal line corresponds to 0.5tr$(P_d P_B)$ + 0.5 tr$(P_B P_B)$ = 0.5 + 0.5tr$(P_d P_B)$. Hence, we take again a 50-50 view of the benchmark model P_B versus the data P_d. The upward sloping line represents the target model pricing error against the classical mixture measurement operator defined by equation (5.9) in Definition 5.1. In panels (a) and (b) of Figure 6.1 we display two target models, both with EIS = 1 and $\gamma = 2$ versus $\gamma = 10$. We observe that the length of the interval between p_L and p_U is unchanged as we increase the parameter γ in the SDF. This result is not surprising given what we know about the challenges pertaining to the estimation of the SDF. Increasing γ in this case does not move the needle so to speak. Note that $P_L = 0.09$, so that the target model is adopted if we have at least 91 % confidence in the data. Conversely, we stick to strong beliefs about the benchmark model in excess of 90 % and therefore very little confidence in the data, we quantum unambiguously disregard the target model.

Panels (c) and (d) modify the classical mix of data and benchmark as a reference point. In panel (c) we begin with little confidence in the data $p = 0.9$. Against that reference, we won't settle quantum unambiguously for the benchmark model when measuring the evidence of the target model since p_U = 0.99. In panel (d) we change to a reference will strong (classical) confidence in the data. Unless, we take a strong view in favor of the benchmark model $p_U = 0.71$, we go for the target model no matter what p we pick, since $p_L = 0$.

We can of course play with p and δ as we proceed to measurement of the target model quantum state and make model decisions guided by expected loss, but we prefer to keep the analysis based on the quantum unambiguous regions outside the $[p_L, p_U]$ interval. One example appeared in panel (c) of Figure 5.1 where we narrowed the range for δ. As the figure illustrated, this obviously affects the length of the $[p_L, p_U]$ interval. In Figure 6.1 we opted not to constraint the range of δ which drives quantum ambiguity.

7 Conclusions and Future Research

Quantum computing hardware is evolving rapidly and this paper is part of the growing interest in its potential applications in economics and finance. While the hardware is not yet up to task for many of the proposed methods presented in this paper, we expect the steady progress of the technology to make the advances we described implementable on real quantum hardware in the foreseeable future. Econometricians will ultimately insert quantum hardware in their research methods, which will enable them to (a) harness quantum ambiguity characterized by superposition and (b) benefit from computational speed-ups along the way. There are still many topics we left untouched which we think are exciting areas of future research. In this concluding section we list a few of them.

The fact that the asset pricing models involve parameters begs for a discussion about the sampling theory suitable for quantum computing algorithms. In principle there is the potential of asymptotic expansions along three dimensions. Two are standard in the econometrics literature, while the third is not - at least at first sight. First a few words about the two familiar ones. There is N, pertaining to the quadrature-based discretization studied by Tauchen and Hussey (1991) with its approximation error diminishing, under suitable regularity conditions, as $N \to \infty$. Unfortunately, making the step to quantum computing means infinite dimensional Hilbert spaces which pose their own challenges. While finite-dimensional and infinite-dimensional Hilbert spaces share some similarities, such as the inner product structure, they also exhibit crucial differences in terms of completeness, basis (e.g. countably infinite), among others. It is beyond the purpose of this paper to explore such extensions. Next is the asymptotics pertaining to the data, in our case underpinning the quantum state $|d\rangle$. A classic textbook discussion regarding asymptotics appears for instance in Silverman (2018).

This bring us to the asymptotic expansion along the third and final dimension: S the number of "shots" or repetitions of the quantum circuit to compute the expectation values appearing in the discussions of Sections 4 and 5.[27] There is a small, but growing, literature on so called quantum statistics. Helstrom (1969) and Holevo (1982) cover the basic foundations, but as Gill (2001) put it: *Quantum statistics mainly consists of exact results in various rather special models* (although written in 2001, it is still largely true). There are some results on MLE of a parameter say θ determining a

[27] There are some similarities with the econometrics literature on simulation-based estimation, see e.g. Duffie and Singleton (1993).

quantum state $|\bar{\nu}(\theta)\rangle$ - for a correctly specified model - and quantum Cramér-Rao lower bounds as $S \to \infty$, see e.g. Barndorff-Nielsen and Gill (2000), Gill (2001), Barndorff-Nielsen et al. (2003), Gill and Guţă (2013), among others. However, much work remains to be done connecting the decision-theoretic foundations of quantum measurement presented in this paper with quantum statistical analysis.

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APPENDIX

A.1 Proof of Theorem 4.1

We start by considering $P_{|\bar{\nu}(\theta)\rangle} = \mathcal{A}^{CD}_{|\bar{\nu}(\theta)\rangle}(0)$, and therefore $\mathbb{A} = P_{\bar{\nu}}$. We need to show that $E[U(P_d, d-\bar{\nu}(\theta))] = E[U(P_{\bar{\nu}}, d-\bar{\nu}(\theta))]$. Using equation (4.1) we need to show that $\text{tr}(P_{d-\bar{\nu}(\theta)}\mathbb{A})$ for $\mathbb{A} = P_d$ and $P_{|\bar{\nu}(\theta)\rangle}$ are equivalent. Let $\mathbb{A} = P_d$, then:

$$\begin{aligned}
\text{tr}(P_d P_{d-\bar{\nu}(\theta)}) &= \langle d-\bar{\nu}(\theta)|\,|d\rangle\,\langle d|\,|d-\bar{\nu}(\theta)\rangle \\
&= \langle d|\,(|d\rangle\,\langle d|)\,|d\rangle - \langle d|\,(|d\rangle\,\langle d|)\,|\bar{\nu}(\theta)\rangle - \langle\bar{\nu}(\theta)|\,(|d\rangle\,\langle d|)\,|d\rangle + \langle\bar{\nu}(\theta)|\,(|d\rangle\,\langle d|)\,|\bar{\nu}(\theta)\rangle \\
&= 1 - 2\text{Re}(\langle d\,|\bar{\nu}(\theta)\rangle) + \|\langle d\,|\bar{\nu}(\theta)\rangle\|^2 \\
&= 1 - 2\text{Re}(\langle d\,|\bar{\nu}(\theta)\rangle) + \text{Re}(\langle d\,|\bar{\nu}(\theta)\rangle)^2 + \text{Im}(\langle d\,|\bar{\nu}(\theta)\rangle)^2 \\
&= (1 - \text{Re}(\langle d\,|\bar{\nu}(\theta)\rangle))^2 + \text{Im}(\langle d\,|\bar{\nu}(\theta)\rangle)^2,
\end{aligned} \tag{A.1}$$

where $\|\langle d\,|\bar{\nu}(\theta)\rangle\| = \sqrt{\langle d\,|\bar{\nu}(\theta)\rangle\,\langle\bar{\nu}(\theta)\,|d\rangle}$ and $\langle d\,|\bar{\nu}(\theta)\rangle = \text{Re}(\langle d\,|\bar{\nu}(\theta)\rangle) + i\text{Im}(\langle d\,|\bar{\nu}(\theta)\rangle)$.

Similarly, let $\mathbb{A} = P_{|\bar{\nu}(\theta)\rangle}$, then:

$$\begin{aligned}
\text{tr}(P_{|\bar{\nu}(\theta)\rangle} P_{d-\bar{\nu}(\theta)}) &= \langle d - \bar{\nu}(\theta) | P_{|\bar{\nu}(\theta)\rangle} | d - \bar{\nu}(\theta)\rangle \\
&= \langle d |\, \bar{\nu}(\theta)\rangle \langle \bar{\nu}(\theta) |\, d\rangle - \langle d |\, \bar{\nu}(\theta)\rangle - \langle \bar{\nu}(\theta) |\, d\rangle + \langle \bar{\nu}(\theta) |\, \bar{\nu}(\theta)\rangle \\
&= 1 - 2\text{Re}(\langle d\,|\bar{\nu}(\theta)\rangle) + \|\langle d\,|\bar{\nu}(\theta)\rangle\,\|^2 \\
&= (1 - \text{Re}(\langle d\,|\bar{\nu}(\theta)\rangle))^2 + \text{Im}(\langle d\,|\bar{\nu}(\theta)\rangle)^2.
\end{aligned} \tag{A.2}$$

Hence $\text{tr}(P_{d-\bar{\nu}(\theta)} P_d) = \text{tr}(P_{d-\bar{\nu}(\theta)} P_{|\bar{\nu}(\theta)\rangle})$ and therefore $E[U(P_d, d - \bar{\nu}(\theta))] = E[U(P_{d-\bar{\nu}}, d - \bar{\nu}(\theta))]$. Next, we consider an act $\mathbb{A} \in \mathcal{A}^{CD}_{|\bar{\nu}(\theta)\rangle}(p)$ for $p \neq 0$. From equation (4.5) we have (recall from Definition 4.2 that $\tilde{\mathbb{A}} \neq P_{|d\rangle}$):

$$\begin{aligned}
\text{tr}(\mathbb{A} P_{d-\bar{\nu}(\theta)}) &= \langle d - \bar{\nu}(\theta) | ((1-p) P_{\bar{\nu}(\theta)} + p\tilde{\mathbb{A}}) | d - \bar{\nu}(\theta)\rangle \\
&= (1-p)\text{tr}(P_{|\bar{\nu}(\theta)\rangle} P_{d-\bar{\nu}(\theta)}) + p\text{tr}(\tilde{\mathbb{A}} P_{d-\bar{\nu}(\theta)}) \\
&\neq \text{tr}(P_{|\bar{\nu}(\theta)\rangle} P_{d-\bar{\nu}(\theta)}), \qquad \text{for } p \neq 0,
\end{aligned} \tag{A.3}$$

since $\tilde{\mathbb{A}} \neq P_{|\bar{\nu}(\theta)\rangle}$, $\tilde{\mathbb{A}} \neq P_{|d\rangle}$ and $\text{tr}(\tilde{\mathbb{A}} P_{d-\bar{\nu}(\theta)}) \neq 0$. □

A.2 Proof of Theorem 5.1

The projection operator can be characterized as follows.[28]

LEMMA A.1. *The projection operator $P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle}$ appearing in equation (5.10) can be written as:*

$$P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle} = \alpha^2 P_1 + (1-\alpha^2) P_2 + \alpha\sqrt{1-\alpha^2}\left(e^{-i\delta} |\mathcal{S}_1\rangle \langle \mathcal{S}_2| + e^{i\delta} |\mathcal{S}_2\rangle \langle \mathcal{S}_1|\right) \tag{A.4}$$

where $P_1 = |\mathcal{S}_1\rangle \langle \mathcal{S}_1|$ and $P_2 = |\mathcal{S}_2\rangle \langle \mathcal{S}_2|$.

Proof: For the projection operator $P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle}$ we have the following:

$$\begin{aligned}
P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle} &= |\mathcal{S}_{1,2}(\alpha,\delta)\rangle \langle \mathcal{S}_{1,2}(\alpha,\delta)| \\
&= \left(\alpha |\mathcal{S}_1\rangle + e^{i\delta}\sqrt{1-\alpha^2} |\mathcal{S}_2\rangle\right)\left(\alpha \langle \mathcal{S}_1| + e^{-i\delta}\sqrt{1-\alpha^2} \langle \mathcal{S}_2|\right) \\
&= \alpha^2 |\mathcal{S}_1\rangle \langle \mathcal{S}_1| + (1-\alpha^2) |\mathcal{S}_2\rangle \langle \mathcal{S}_2| + \alpha\sqrt{1-\alpha^2}\left(e^{-i\delta} |\mathcal{S}_1\rangle \langle \mathcal{S}_2| + e^{i\delta} |\mathcal{S}_2\rangle \langle \mathcal{S}_1|\right)
\end{aligned}$$

The result in equation (A.4) follows. □

Next we turn to the proof of Theorem 5.1. For $P_3 = |\mathcal{S}_3\rangle \langle \mathcal{S}_3|$, $P_1 |\mathcal{S}_3\rangle = c_1^3 |\mathcal{S}_1\rangle$ with $c_1^3 = \langle \mathcal{S}_1 | \mathcal{S}_3\rangle$, and $P_1 |\mathcal{S}_2\rangle =$

[28] We omit the proof of Lemma A.1 as it simply follows from applying Euler's formula for $e^{i\delta}$.

$c_1^2 |\mathcal{S}_1\rangle$ with $c_1^2 = \langle \mathcal{S}_1 | \mathcal{S}_2 \rangle$, we have:

$$\begin{aligned}
\text{tr}(P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle} P_3) &= \text{tr}\left(\alpha^2 P_1 + (1-\alpha^2)P_2 + \alpha\sqrt{1-\alpha^2}\left(e^{-i\delta}|\mathcal{S}_1\rangle\langle\mathcal{S}_2| + e^{i\delta}|\mathcal{S}_2\rangle\langle\mathcal{S}_1|\right)\right) P_3 \\
&= \alpha^2 \text{tr}\left(P_1 P_3\right) + (1-\alpha^2)\text{tr}\left(P_2 P_3\right) \\
&\quad + \alpha\sqrt{1-\alpha^2}\text{tr}\left(e^{-i\delta}|\mathcal{S}_1\rangle\langle\mathcal{S}_2| P_3 + e^{i\delta}|\mathcal{S}_2\rangle\langle\mathcal{S}_1| P_3\right) \\
&= \alpha^2 \text{tr}\left(P_1 P_3\right) + (1-\alpha^2)\text{tr}\left(P_2 P_3\right) \\
&\quad + \alpha\sqrt{1-\alpha^2}\text{tr}\left(e^{-i\delta}|\mathcal{S}_1\rangle\langle\mathcal{S}_2| P_3 + \overline{e^{-i\delta}|\mathcal{S}_1\rangle\langle\mathcal{S}_2| P_3}\right) \\
&= \alpha^2 \text{tr}\left(P_1 P_3\right) + (1-\alpha^2)\text{tr}\left(P_2 P_3\right) \\
&\quad + \alpha\sqrt{1-\alpha^2}\text{tr}\left(e^{-i\delta}|\mathcal{S}_1\rangle\langle\mathcal{S}_2| P_3\right) + \alpha\sqrt{1-\alpha^2}\text{tr}\left(\overline{e^{-i\delta}|\mathcal{S}_1\rangle\langle\mathcal{S}_2| P_3}\right) \\
&= \alpha^2 \text{tr}\left(P_1 P_3\right) + (1-\alpha^2)\text{tr}\left(P_2 P_3\right) \\
&\quad + \alpha\sqrt{1-\alpha^2}\left[\text{tr}\left(e^{-i\delta}|\mathcal{S}_1\rangle\langle\mathcal{S}_2| P_3\right) + \overline{\text{tr}\left(e^{-i\delta}|\mathcal{S}_1\rangle\langle\mathcal{S}_2| P_3\right)}\right].
\end{aligned}$$

Furthermore:

$$\begin{aligned}
\left[\text{tr}\left(e^{-i\delta}|\mathcal{S}_1\rangle\langle\mathcal{S}_2| P_3\right) + \overline{\text{tr}\left(e^{-i\delta}|\mathcal{S}_1\rangle\langle\mathcal{S}_2| P_3\right)}\right] &= 2\text{Re}\left[\text{tr}\left(e^{-i\delta}|\mathcal{S}_1\rangle\langle\mathcal{S}_2| P_3\right)\right] \\
&= 2\text{Re}\left[\text{tr}\left(e^{-i\delta}|\mathcal{S}_1\rangle\langle\mathcal{S}_2| P_3 P_3\right)\right] \\
&= 2\text{Re}\left[\text{tr}\left(e^{-i\delta} P_3 |\mathcal{S}_1\rangle\langle\mathcal{S}_2| P_3\right)\right] \\
&= 2\text{Re}\left[\text{tr}\left(e^{-i\delta} c_3^1 |\mathcal{S}_3\rangle \bar{c}_3^2 \langle\mathcal{S}_3|\right)\right] \\
&= 2\text{Re}\left[\text{tr}\left(e^{-i\delta} c_3^2 \bar{c}_3^1 |\mathcal{S}_3\rangle\langle\mathcal{S}_3|\right)\right] \\
&= \text{tr}\left(\text{Re}\left[e^{-i\delta} c_3^2 \bar{c}_3^1 |\mathcal{S}_3\rangle\langle\mathcal{S}_3|\right]\right) \\
&= \text{Re}\left[e^{-i\delta} c_3^2 \bar{c}_3^1\right] \text{tr}\left(|\mathcal{S}_3\rangle\langle\mathcal{S}_3|\right) \\
&= \text{Re}\left[e^{-i\delta} c_3^2 \bar{c}_3^1\right] \qquad \text{since tr} P_3 = 1,
\end{aligned}$$

where $c_3^1 = \langle \mathcal{S}_3 | \mathcal{S}_1 \rangle$, $c_3^2 = \langle \mathcal{S}_3 | \mathcal{S}_2 \rangle$, and hence $\bar{c}_3^2 = \langle \mathcal{S}_2 | \mathcal{S}_3 \rangle$. Moreover, let $c_3^1 = r_3^1 e^{i\alpha_3^1}$ and $c_3^2 = r_3^1 e^{i\alpha_3^1}$. We can then write:

$$\begin{aligned}
\left[\text{tr}\left(e^{-i\delta}|\mathcal{S}_1\rangle\langle\mathcal{S}_2| P_3\right) + \overline{\text{tr}\left(e^{-i\delta}|\mathcal{S}_1\rangle\langle\mathcal{S}_2| P_3\right)}\right] &= \text{Re}\left[e^{-i\delta} r_3^2 e^{-i\alpha_3^2} r_3^1 e^{i\alpha_3^1}\right] = r_3^1 r_3^2 \text{Re}\left[e^{-i(\delta+\alpha_3^2-\alpha_3^1)}\right] \\
&= r_3^1 r_3^2 \cos(\delta + \alpha_3^2 - \alpha_3^1) \\
&= r_3^1 r_3^2 \cos(\delta + \alpha_3^{1,2}),
\end{aligned}$$

where $\alpha_3^{1,2} = \alpha_3^2 - \alpha_3^1$. Collecting all the terms we have:

$$\text{tr}(P_{|\mathcal{S}_{1,2}(\alpha,\delta)\rangle} P_3) = \alpha^2 \text{tr}\left(P_1 P_3\right) + (1-\alpha^2)\text{tr}\left(P_2 P_3\right) + (\alpha\sqrt{1-\alpha^2}) r_3^1 r_3^2 \cos(\delta + \alpha_3^{1,2}) \tag{A.5}$$

□

A.3 Dynamic Asset Pricing Models - Technical Details

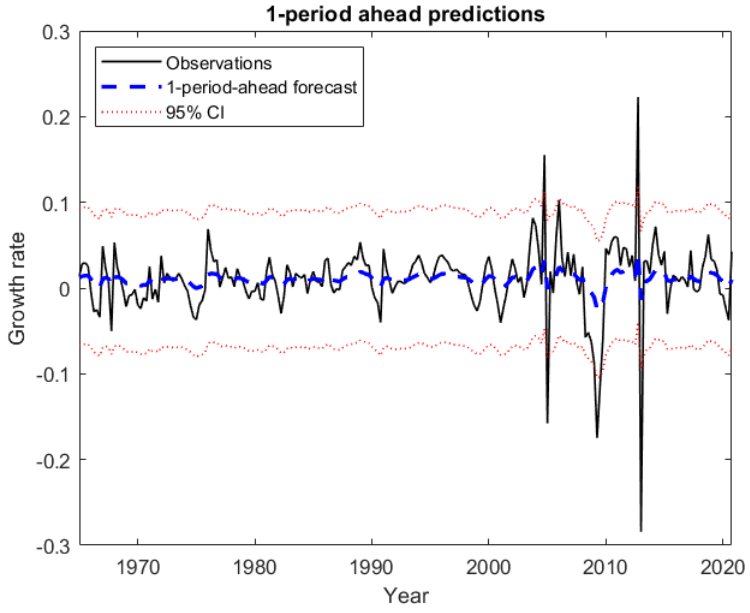
A.3.1 Dividend Growth Model

We start with equation (2) of Hansen et al. (2008) applied to log dividend growth instead of consumption. We download data from 1964Q1 to 2020Q4 from FRED (Personal income receipts on assets: Personal dividend income (series: B703RC1Q027SBEA), which is seasonally adjusted) and then deflate using CPI growth. Then log real dividend growth is modeled as follows:

$$\begin{aligned} d_{t+1} - d_t &= a + x_{t+1} + be_{t+1} \\ x_{t+1} &= \rho_1 x_t + c\epsilon_{t+1}, \end{aligned} \tag{A.6}$$

where the shocks $e_{t+1} \sim N(0,1)$ and $\epsilon_{t+1} \sim N(0,1)$.

Figure A.1: Observations and predictions with 95% confidence intervals

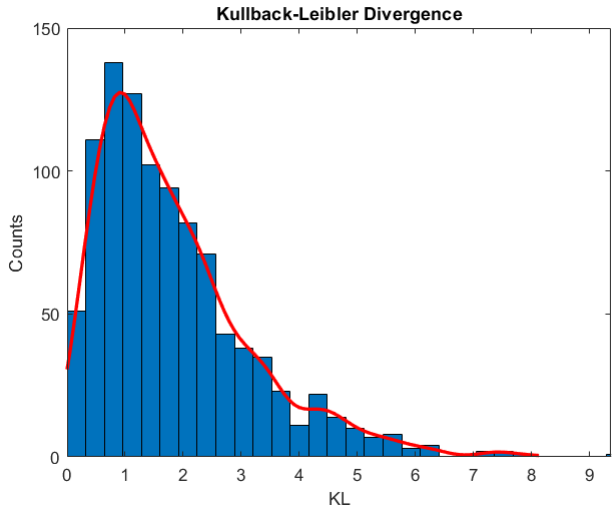


We collect the parameters of the above model into the vector $\theta_L = (a, b, c, \rho_1)'$. We can estimate the parameters of the model (A.6) via Maximum Likelihood. To estimate the model we use data covering the sample 1964Q1 to 2020Q4 from FRED (Personal income receipts on assets: Personal dividend income (series: B703RC1Q027SBEA), which is seasonally adjusted) and then deflate using CPI growth. Table A.1 reports the parameter estimates and their standard errors. Note that all the parameters are statistically significant. The one-period ahead mean predictions appear in Figure A.1. During the sample we observe some extreme observations, mostly related to the financial crisis, which are not well captured by the model.

Table A.1: Parameter estimates for log real dividend growth

Parameter	Coefficient	Standard Error
ρ_1	0.64079	0.25901
c	0.01520	0.00901
b	0.03630	0.00272
a	0.01037	0.00416

Figure A.2: Distribution of Kullback-Leibler divergence from asymptotic distribution random draws



We rely on the asymptotic distribution theory for MLE to characterize model uncertainty about equation (A.6). Namely, we assume that the model is correctly specified, but we are uncertain about its parameter values. We create one thousand draws of parameter models $\{\theta_L^i, i = 1, \dots 1000\}$ and construct a 1000 models for log dividend growth. More specifically, we draw one thousand $\theta_L^i \sim N(\hat{\theta}_L, \hat{\Sigma})$, excluding draws with $\rho_1 < 0$ and $\rho_1 > 1$. The collection of models therefore represents statistical uncertainty, since we assume the model specification is correct and asymptotic distribution theory provides us guidance about parameter uncertainty. We can characterize and visualize the uncertainty we calculate the Kullback-Leibler divergence $KL_i = \frac{1}{2}[(\theta_L^i - \hat{\theta}_L)'\hat{\Sigma}^{-1}(\theta_L^i - \hat{\theta}_L)]$. The distribution appears in Figure A.2.

A.3.2 Stochastic Discount Factor

Next, we specify the log of the one period stochastic discount factor $s_{t+1,t}$:

$$s_{t+1,t} = \alpha_0 + \alpha_1 x_t + \xi w_{t+1}. \tag{A.7}$$

We know that for the risk-free rate r_t^f :

$$\frac{1}{r_t^f} = \mathbb{E}_t \exp\left[\alpha_0 + \alpha_1 x_t + \xi w_{t+1}\right] = \exp\left[\alpha_0 + \alpha_1 x_t\right] \mathbb{E}_t \exp\left[\xi w_{t+1}\right] = \exp\left[\alpha_0 + \alpha_1 x_t\right]$$

since w_{t+1} is i.i.d. with mean zero. Assuming $r_t^f > 0 \; \forall \; t$ (implied by no arbitrage) and taking a sample average, we get:

$$T^{-1} \sum_{t=1}^{T} - \log r_t^f = T^{-1} \sum_{t=1}^{T} [\alpha_0 + \alpha_1 x_t] = \hat{\alpha}_0,$$

since x_t is a mean zero process. This yields an estimate for α_0. We use sample average of the risk-free rate - 3-month T-Bill from same sample period of 1964Q1 to 2020Q4.[29] To obtain an estimate for α_1 we note that:

$$- \log r_t^f - \hat{\alpha}_0 = \alpha_1 x_t = \alpha_1 [\hat{x}_t - \epsilon_t]$$

where $\hat{x}_t$ are the Kalman filtered state estimates with $\hat{x}_t \perp \epsilon_t$, where ϵ_t is the filtering error.[30] This means that we can recover an estimate $\hat{\alpha}_1$ from running a regression of demeaned minus log risk-free rates onto filtered $\hat{x}_t$. Given our sample, the numerical values are as follows: $\hat{\alpha}_0$ = -0.8974, $\hat{\alpha}_1$ = 1.2038.

Next, we compute with the implied value with β = 0.99 the following (using the formula middle of page 264 of Hansen et al. (2008)):

$$\begin{aligned}
\xi &= -\gamma b + (1-\gamma)\beta [1 \; 0 \; 0 \; 0] \left[I - \beta \times \begin{bmatrix} \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right]^{-1} \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= -\gamma b + (1-\gamma)\beta [1 \; 0 \; 0 \; 0] \begin{bmatrix} 1-\beta\rho_1 & -\beta\rho_2 & -\beta\rho_3 & -\beta\rho_4 \\ -\beta & 1 & 0 & 0 \\ 0 & -\beta & 1 & 0 \\ 0 & 0 & -\beta & 1 \end{bmatrix}^{-1} \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\gamma b + \frac{(1-\gamma) c \beta}{1 - \sum_{i=1}^{4} \beta^i \rho_i}
\end{aligned}$$

[29] Data is downloaded from FRED (`https://fred.stlouisfed.org/series/DTB3`) in percentage terms, and we take quarterly average of all daily yield.

[30] The Kalman filter is implemented in MATLAB. The initial state vector and covariance matrix are both 0.

We have estimates for b, c, β, and ρ_i, so that we can compute ξ as a function of γ :

$$\begin{aligned}\xi &= \frac{c\beta}{1-\beta\rho_1} - \left(\frac{c\beta}{1-\beta\rho_1} + b\right)\gamma \\ &= 0.0412 - 0.0775\gamma \qquad \text{Using estimated parameter values.}\end{aligned}$$

Next we provide the details of the practical implementation for a single model, which means we select a given $\theta = (\theta_L, \gamma)$. We set $N = 4$, or more precisely we compute the quadrature approximation to the AR(1) model for the log dividend growth appearing in equation (A.6) with 4 abscissa using the code of Farmer and Toda (2017) matching the first two sample moments of the data.[31] This yields the transition matrix $\Pi_4^x(\theta_L)$, using the notation of equation (2.6). The log of the SDF in equation (A.7) involves a standard Gaussian shock w_{t+1}. We assume a one-standard deviation positive/negative shock, i.e. $w_{t+1} = \pm 1$, reflecting a "good" versus "bad" economic environment. Since the shock is assumed to be independent of x_t, we have the following transition density: $\Pi_{16}(\theta) = \Pi_4^x(\theta_L) \otimes_K S_2$, where $\otimes_K$ is a Kronecker product and S_2 is the two-dimensional matrix where all entries are equal to 1/2. Recall that we need to invert the matrix $\mathcal{C}_{16}(\theta) = \mathcal{B}_{16}^{-1}(\theta)\mathcal{A}_{16}(\theta)$ where:

- $\mathcal{A}_{16}(\theta) = [I_{16} - \mathcal{H}_{16}(\theta_L) \circ \mathcal{M}_{16}(\theta) \circ \Pi_{16}(\theta_L)]$
- $\mathcal{B}_{16}(\theta) = \text{diag}(\sqrt{16}\sum_{k=1}^{16} \psi_{1k}\pi_{1k}, \ldots, \sqrt{16}\sum_{k=1}^{16} \psi_{16k}\pi_{16k})$.

To that end, we first define the following objects:

- $x_i(\theta_L)$ for $i = 1, \ldots, 8$ are the abscissa for the quadrature discretization of the AR(1) model
- As already noted, $\Pi_{16}(\theta_L) = [\pi_{ij}]_{i,j=1,\ldots,16} = \Pi_8(\theta_L) \otimes_K S_2$.
- $\mathbb{S}_8(\theta_L) = [\exp(-0.8970 + 1.2038x_i(\theta_L))]_{i=1,\ldots,8}$, a 8×1 vector, using equation (A.7)
- $\mathcal{M}_{16}(\theta) = [m_{ij}]_{i,j=1,\ldots,16} = [\mathbb{S}_8(\theta_L)\mathbf{1}_8'] \otimes_K \begin{pmatrix} \exp(\xi) & \exp(-\xi) \\ \exp(\xi) & \exp(-\xi) \end{pmatrix}$ where $\mathbf{1}_8$ is a 8×1 vector of ones. Note that for the CRRA utility function $\xi = -b\gamma = -0.03630\gamma$, using the parameter estimates reported in Table A.1, and $\xi =$ 0.0412 - 0.0775γ for recursive utility with the intertemporal elasticity of substitution equal to one.
- We set γ = 2, 5 and 10, which yields a total of six SDF specifications.
- Using equation (A.6) we construct dividend growth: $\mathbb{D}_8(\theta_L) = [\exp(0.01037 + x_i(\theta_L))]_{i=1,\ldots,8}$, and compute $\mathcal{H}_{16}(\theta_L) = [\mathbb{D}_8(\theta_L)\mathbf{1}_8'] \otimes_K \begin{pmatrix} \exp(0.03630) & \exp(-0.03630) \\ \exp(0.03630) & \exp(-0.03630) \end{pmatrix}$
- $[\psi_{ij}]_{i,j=1,\ldots,16} = \mathcal{H}_{16}(\theta_L) \circ \mathcal{M}_{16}(\theta)$

With the above, we have all the elements to compute $\mathcal{B}_{16}^{-1}(\theta)$ and $\mathcal{A}_{16}(\theta)$.

[31] The code is available at `https://github.com/alexisakira/discretization` .